An Admissible Macro-Finance Model of the US Treasury Market

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This paper develops a macro-finance model of the yield curve and uses this to explain the behavior of the US Treasury market. Unlike previous macro-finance models which assume a homoscedastic error process and suppose that the one-period return is directly observable, I develop a general affine model which relaxes these assumptions. My empirical specification uses a single conditioning factor and is thus the macro-finance analogue of the $EA_1(N)$ specification of the mainstream finance literature. This model provides a decisive rejection of the standard $EA_0(N)$ macro-finance specification. The resulting specification provides a flexible 10-factor explanation of the behavior of the US yield curve, keying it in to the behavior of the macroeconomy. (JEL: C13, C32, E30, E44, E52)

I. Introduction

Macro-finance models use both observable macroeconomic and unobservable latent variables to model the macroeconomy and bond market, in contrast to the conventional approach which only uses latent variables. Like the conventional approach, it describes yields as linear functions of these driving variables in a way that removes arbitrage opportunities. This new approach allows the parameters of the model to be informed by both macroeconomic and yield data. It generates models that are easier to interpret and understand since they are based upon

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standard macroeconomic structures. However, the current macro-finance specification suffers from a number of drawbacks compared to the conventional one. In particular it assumes that the volatility structure is constant, while the conventional literature finds that square root volatility is significant. Also, macro-finance modelers assume that the interest rate that is relevant for the yield structure can be identified and observed without error, while the conventional finance model estimates this as a linear combination of the latent factors. In this paper I develop a model of the US economy and Treasury bond market which relaxes these assumptions, bridging the gap between the macro-finance and conventional models of the term structure.

Since these various models are linear in variables, this means that if there are N underlying driving variables, they can be represented by N bond yields or macroeconomic variables or both. For example the conventional 'yield factor' approach just uses linear combinations of N bond yields as factors assuming that these are observed without measurement error. It has been extensively used for testing affine specifications (Brown and Schaefer (1994), Duffie and Kan (1996), Dai and Singleton (2002)). Macro-finance models on the other hand are based on the 'central bank model' (CBM) developed by Svensson (1999); Smets (1999) and others. This represents the behavior of the macroeconomy in terms of three variables: inflation (π_t), the gap between output and its inflation-neutral level (g_t) and a policy interest rate like the Fed Funds rate (r_t). This provides a basic dynamic description of an economy in which the central bank implicitly targets inflation using a 'Taylor rule', which determines the policy rate in terms of inflation and the output gap. Early macro-finance papers (Ang and Piazzesi (2003)) revealed that the CBM provides a good description of the behavior of short term yields but that a latent variable known as the 'financial factor' has to be added to explain long term yields. Consequently, my model employs the three macroeconomic variables of the CBM with two sets of lags, together with a single latent variable representing the financial factor. This gives a total of N = 10 state variables. The financial factor is backed out from the yield model as a linear combination of the 9 observable variables and the 15 year yield.

I depart from the macro-finance literature in assuming that both the mean values and variances of the system are linear in the financial factor. This means that the yield curve is determined by the square root volatility model of Cox et al (1985). To handle quarterly economic data I employ the discrete time version of this model developed by Sun

(1992). In order to ensure that the variance structure remains non-negative, I also employ 'admissibility' restrictions similar to those proposed for the continuous time model by Dai and Singleton (2002). This is the analogue of the (EA_1) yield factor model developed by Dai and Singleton (2002) and Dai and Singleton (2002), which as they say: 'builds upon a branch of the finance literature that posits a short-rate process with a single stochastic central tendency and volatility'. Despite the extensive use of stochastic volatility models in theoretical and empirical finance papers and the evidence of heteroscedasticity in macroeconomic and asset price data this is the first macro-finance model with this feature. Finally I follow the conventional finance approach in assuming that the one-period yield or 'spot rate' relevant to the term structure $\left(y_{1,t}\right)$ is a linear combination of the state variables and not necessarily equal to the 'policy' interest rate (r_t) generated by the macro model. These innovations significantly improve the explanatory power of the macro-finance model and provide further insights into the working of the US economy and bond market.

The paper is set out along the following lines. The next section develops a Vector Auto-Regression (VAR) model of the economy and section III shows how this can be used to derive an affine term structure under the no-arbitrage assumption. Section IV then compares the performance of my models against the standard macro-finance model and discusses the implications for the economy and bond market. Section V offers a brief conclusion.

II. The Macroeconomic Framework

My model represents the behavior of the macroeconomy in terms of the annual CPI inflation rate (π_t), output gap (g_t) and the 3 month Treasury Bill rate (r_t). These form the vector $z_t = \{\pi_t, g_t, r_t\}$ of macroeconomic variables. This T-bill rate is chosen as the 'policy rate' in preference to alternatives like the Federal Funds and Euro-dollar rates since it has a 3 month maturity and is default free, likely to make it more relevant to a quarterly model of the Treasury market. In addition, $x_{1,t}$ represents the financial factor. This is assumed to follow the first order autoregressive process:

$$x_{1,t+1} = \theta + \xi x_{1,t} + w_{1,t+1} \tag{1}$$

where $w_{1,t+1}$ is an equation residual or error defined in the next subsection and z_t is driven by the $L - th$ order difference system:

$$z_{t+1} = \kappa + \phi_0 x_{1,t} + \sum_{l=1}^{L} \phi_l z_{t+1-l} + w_{2,t+1} \tag{2}$$

where $w_{2,t}$ is an error vector. These are decomposed into components that are related to $w_{1,t}$ and an orthogonal component η_t:

$$w_{2,t+1} = Hw_{1,t+1} + G\eta_{t+1} \tag{3}$$

This system is consolidated by defining $x_t = \{x_{1,t}, z_t'\}'$; $w_t = \{w_{1,t}, w_{2,t}'\}'$; $\upsilon_t = \{w_{1,t}, \eta_t'\}'$; and combining (1) and (2), to give an L–th order difference system for n stochastic variables:[1]

$$x_{t+1} = \hat{x}_{t+1} + w_{t+1} \tag{4}$$

$$\text{where:} \quad \hat{x}_{t+1} = K + \sum_{l=1}^{L} \Gamma_l x_{t+1-l}$$

and:

$$w_t = \Gamma \upsilon_t; \quad K = \begin{bmatrix} \theta \\ \kappa \end{bmatrix}; \quad \Gamma = \begin{bmatrix} 1 & 0_{1,3} \\ H & G \end{bmatrix};$$

$$\Gamma_1 = \begin{bmatrix} \xi & 0_{1,3} \\ \Phi_0 & \Phi_1 \end{bmatrix}; \quad \Gamma_l = \begin{bmatrix} 0 & 0_{1,3} \\ 0_{3,1} & \Phi_l \end{bmatrix}; l = 2, ..., L.$$

In this paper, a hat over any variable like $\hat{x}_{t+1}$ indicates its conditional expectation in the previous period. The yield model employs the state space form, obtained by arranging this as a first order difference system describing the dynamics of the state vector (see appendix 1):

1. In this paper, Diag{y} represents a matrix with the vector y in the diagonal and zeros elsewhere. 0_a is the ($a \times 1$) zero vector; 1_a is the ($a \times 1$) summation vector; $0_{a,b}$ the ($a \times b$) zero matrix; I_a the a^2 identity matrix. and $I_{a,b}$ an a^2 matrix with ones in the first b elements of the leading diagonal and zeros elsewhere.

$$X_t = \Theta + \Phi X_{t-1} + W_t \tag{5}$$

where $X_t = \left\{x_{1,t}, z_t' \ ,..., z_{t-L}'\right\}'$ is the state vector, $W_t = C.\left\{w_t' \ ,0_{1,N-n}\right\}'$ and Θ, Φ & C are defined in appendix 1. X_t has dimension $N = 1+3L = 10$. Similarly, writing $X_t = \left\{x_{1,t}, X_{2,t}'\right\}'$ and partitioning W_t, Θ, Φ, C conformably (see appendix 1), (5) becomes:

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} \theta \\ \Theta_2 \end{bmatrix} + \begin{bmatrix} \xi & 0_{N-1}' \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} w_{1,t+1} \\ W_{2,t+1} \end{bmatrix} \tag{6}$$

The macroeconomic data were provided by Datastream and are shown in figure 1. π_t is the annual CPI inflation rate and r_t the 3 month Treasury Bill rate. The output gap series g_t is the quarterly OECD measure, derived from a Hodrick-Prescott filter. The yield data were taken from McCulloch and Kwon (1991), updated by the New York Federal Reserve Bank.[2] These have been extensively used in the empirical literature on the yield curve. The 15 year yield (the longest for which a continuous series is available) is used in equation (16) below to identify the financial factor. The 1,2,3,5,7, and 10 year yields are modelled as dependent variables. The macroeconomic data dictated a quarterly time frame (1961Q4-2004Q1, a total of 170 periods). The quarterly yield data are shown in figure 2. The 15 year yield is shown at the back of the figure, while the shorter maturity yields are shown at the front.

A. The Stochastic Structure

The standard macro-finance model assumes that the volatility structure is homoscedastic and Gaussian: $W_{t+1} \sim N\left(0_N, \Omega\right)$. However, conventional finance models usually assume that volatility is stochastic. In the affine model developed by Duffie and Kan (1996) and Dai and Singleton (2002), conditional heteroscedasticity in the errors is driven by square root processes in the state variables. In the 'admissible' version of this specification developed by Dai and Singleton (2002),

2. I am grateful to Tony Rodrigues of the New York Fed for supplying a copy of this yield dataset.

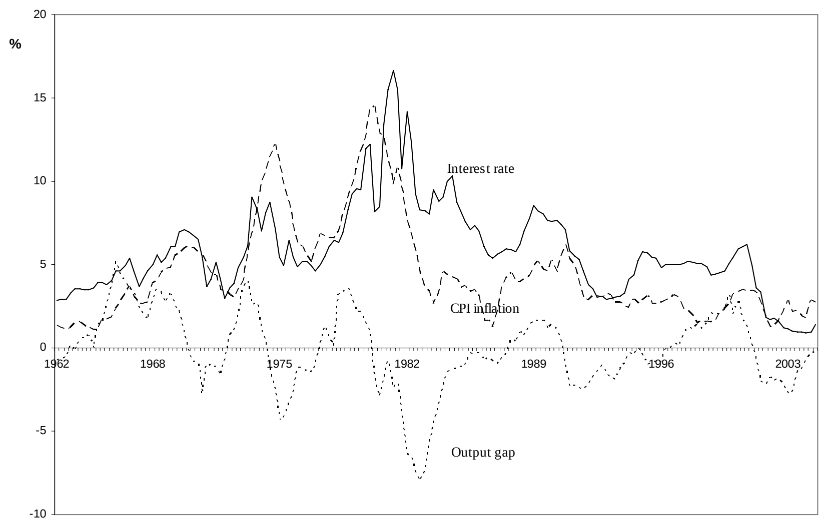


FIGURE 1.— Macroeconomic variables

Note: CPI Inflation and 3 month T-bill interest rate are from Datastream. Output gap is from OECD.

regularity or admissibility conditions are imposed to ensure that the variance structure remains non-negative definite. Variations in the risk premia depend entirely upon variations in volatility in these models. In the 'Essentially Affine' model of Duffee (2002) state variables can affect risk premia through the price of risk as well as through volatility. In the notation of Dai and Singleton (2002) an admissible essentially affine model with N state variables and m independent square root factors conditioning volatility is classed as EA_m (N). Thus the standard macrofinance model (which is 'essentially affine' and homoscedastic) is denoted $EA_0(N)$. There may in general be several stochastic volatility terms, but in this paper I assume that $m = 1$: stochastic volatility is conditioned by a single variable $x_{1,t}$ that follows a discrete time process represented by (1) with:

$$w_{1,t+1} = u_{1,t+1}\sqrt{\delta_{10} + \delta_{11}x_{1,t}}; \tag{7}$$

where $u_{1,t+1}$ is a standardized normal *i.i.d.* error term. Setting $\delta_{11} = 0$; $\delta_{10} > 0$ gives my Vasicek (1977) equivalent EA_0 model. Setting $\delta_{11} > 0$; δ_{10}

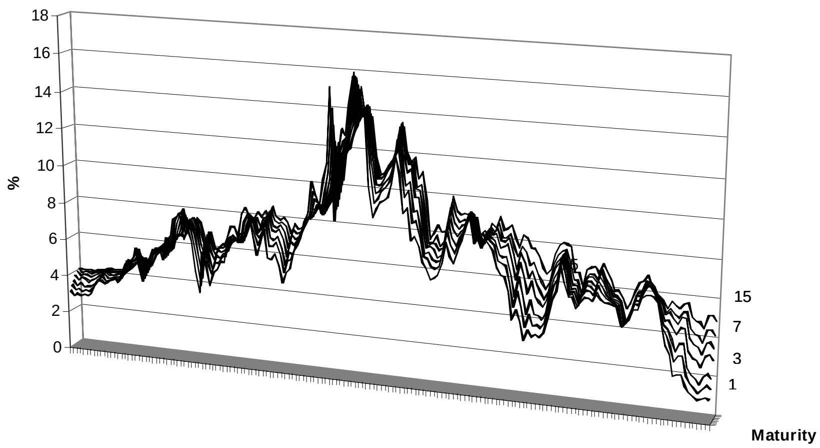


FIGURE 2.— US Treasury discount yields

Note: US Treasury discount bond equivalent yield data compliled by McCulloch and Kwon (1990) updated by the NY Fed.

= 0 gives the discrete time Cox et al (1985) equivalent specification of Sun (1992) and Campbell et al (1996). In this model, volatility disappears as $x_{1,t}$ falls to zero and with θ > 0, mean reversion helps to ensure that $x_{1,t+1} > 0$. This means that the probability of a negative value of variance term is very small and goes to zero in the continuous time limit. If $\delta_{10} \neq 0$ the model generalizes to that of Duffie and Kan (1996). In this model volatility disappears as $\delta_{10} + \delta_{11} x_1$ falls to zero and x_1 falls to $x_{\min} = (-)\ \delta_{10}/\delta_{11}$. Provided that $\theta > (1 - \xi)x_{\min}$ then mean reversion means that the probability of a negative value of variance term (i.e. the probability that x_1 falls below $x_{\min}$) is very small. This is the basic specification of my EA_1 model.

In EA_1, this financial factor also conditions the volatility of the other variables. Substituting $\eta_t = s_t u_{2,t}$ into (3), where $u_{2,t}$ is a vector of standard normal normal *i.i.d*. error terms:

$$w_{2,t+1} = H w_{1,t+1} + G s_t u_{2,t+1} \tag{8}$$

where:

$$E\left[u_{2,t+1} w_{1,t+1}\right] = 0_3;$$

$$s_t = Diag\left\{\left(\delta_{20} + \delta_{21}x_{1,t}\right)^{\frac{1}{2}}, ..., \left(\delta_{40} + \delta_{41}x_{1,t}\right)^{\frac{1}{2}}\right\}.$$

and δ_{i0}, $\delta_{i1} \geq 0$, $i = 2, 3, 4$ and the components of $u_{2,t+1}$ are standardized normal variables. It follows from (4) that:

$$\upsilon_{t+1} \sim N\left(0_n, \Delta_t\right); \quad w_{t+1} \sim N\left(0_n, \Gamma\Delta_t\Gamma'\right); \tag{9}$$

where: $\Delta_t = \Delta_0 + \Delta_1 x_{1,t}$; $\Delta_j = Diag\{\delta_{1j}, ..., \delta_{4j}\}$; $j = 0,1$. The stochastic structure for (6) is described in appendix 1. To make the EA_1 model admissible in the sense of Dai and Singleton (2002), the estimation program checks that:

$$\delta_{i0} / \delta_{10} > \delta_{i1} / \delta_{11}, i = 2,3,4 \tag{10}$$

ensuring that $\delta_{i0} + \delta_{i1} x_{\min} > 0$. This keeps the elements of s_t and hence the variance structure non-negative. One implication of this restriction is that δ_{i1}; $i = 2, 3, 4$ must go to zero with δ_{11} making the structure entirely homoscedastic.

III. The Bond Pricing Framework

The aim of this paper is to use this framework to model the macroeconomy and the yield curve jointly. The macro model is defined under the state probability measure *P*, but assets are priced under the risk neutral measure *Q*. This adjusts the state probabilities in such a way that all assets have the same expected return.

A. The Pricing Kernel

Discount bond prices are obtained using the pricing kernel (Campbell et al (1996), Cochrane (2002)):

$$P_{\tau,t} = \exp\left\{-y_{1,t}\right\} E_t^Q \left[P_{\tau-1,t+1}\right]$$

$$= E_t\left[M_{t+1} P_{\tau-1,t+1}\right]; \ \tau = 1, ..., M. \tag{11}$$

where $P_{\tau,t}$ is the t-period price and E_t^Q denotes the conditional expectation under the risk neutral measure Q. M_{t+1} is the nominal Stochastic Discount Factor (SDF) which changes the probability measure from P to Q and applies the time discount using: $y_{1,t}$. This is known as the 'spot rate': the one-period yield relevant to the term structure and is assumed to be a linear combination of the state variables:

$$y_{1,t} = j_1 x_{1,t} + J_2' X_{2,t}. \tag{12}$$

where J_2 is a 9 × 1 vector. Following the conventional finance approach, these can consist of 9 freely estimated weights. In the standard macrofinance model these weights are restricted to pick out the current value of the policy interest rate r_t from the state vector $\left(y_{1,t} = r_t = J_2' X_{2,t}\right)$ by setting the third element of J_2 to unity and the other weights to zero. Tests of this restriction are reported in the next section.

The logarithm of the SDF is $m_{t+1} = -\left(\omega_t + y_{1,t} + \lambda_{1,t} w_{1,t+1} + \lambda_{2,t}' \upsilon_{2,t+1}\right)$, where $\lambda_{1,t}$ is a scalar and $\lambda_{2,t}$ a 3 × 1 vector of coefficients related to the prices of risk associated with shocks to z_{t+1} . For the yield model to be affine these coefficients must also be affine in the state variables. So for example, the variable $\lambda_{1,t}$ shows the price of risk associated with the financial factor, which plays an important role in this analysis and is specified as:

$$\lambda_{1,t} = \lambda_{10} + \lambda_{11} x_{1,t}. \tag{13}$$

If this is zero, then a portfolio that is constructed so that it is only exposed to shocks in $x_{1,t}$ has a zero risk premium and is expected to earn the spot rate. If it is constant $\lambda_{1,t} = \lambda_{10}$, then variations in this risk premium depend only upon variations in volatility, such as those induced by $x_{1,t}$ in EA_1. This parameter plays the key role in that model. If λ_{11} is also non-zero then this factor can influence the risk premia thorough variations in the price of risk, even if volatility is fixed as in EA_0, so λ_{11} plays the key role in that model. Appendix 2 shows how the prices of risk associated with the other variables are adjusted, following Duffee (2002).

B. Affine Yield Models

Appendix 2 shows that these specifications generate an exponential affine bond price (affine yield) model:

$$P_{\tau,t} = \exp[-\gamma_\tau - \Psi'_\tau X_t]; \quad \tau = 1, ..., M. \tag{14}$$

where Ψ_τ is partitioned conformably with (6) as $\Psi_\tau = \{\psi_{1,\tau}, \Psi'_{2,\tau}\}'$. Taking logs, reversing sign and dividing by maturity τ gives the discount yield:

$$y_{\tau,t} = -p_{\tau,t} / \tau \tag{15}$$

$$= a_\tau + \beta'_\tau X_t + e_{\tau,t}$$

$$= a_\tau + \beta'_{\tau,0} x_{1,t} + \sum_{l=1}^{L} \beta'_{\tau,l} z_{t+1-l} + e_{\tau,t}$$

where: $a_\tau = \gamma_\tau / \tau$; $\beta_\tau = \Psi_\tau / \tau$; and $e_{\tau,t}$ is an *i.i.d*. error. The slope coefficients of the yield system β_τ are known as 'factor loadings.' The standard assumption is that $e_{\tau,t}$ represents measurement error which is homoscedastic and orthogonal to the errors W_t in the macroeconomic system (5).

Following the yield factor approach, I assume that the 15 year (60 quarter) maturity yield $y_{60,t}$ is measured without error: $y_{60,t} = a_{60} + \beta_{60,0} x_{1,t} + \sum_{l=1}^{L} \beta'_{60,l} z_{t+1-l}$. This allows the financial factor to be backed out from the system as:

$$x_{1,t} = \beta_{60,0}^{-1} \left(y_{60,t} - a_{60} - \sum_{l=1}^{L} \beta'_{60,l} z_{t+1-l} \right). \tag{16}$$

Stacking (15) for the 1,2,3,5, 7 and 10 year maturities that are modelled gives a multivariate system for $y_t = \{y_{4,t}, y_{8,t}, y_{12,t}, y_{20,t}, y_{28,t}, y_{40,t},\}$:

$$y_t = a + \beta_0 x_{1,t} + \sum_{l=1}^{L} \beta'_l z_{t+1-l} + e_t; \tag{17}$$

$$e_t \sim N(0, \bar{P}); \quad \bar{P} = Diag\{\rho_1, \rho_2, ..., \rho_6\}.$$

This defines the yields in terms of the current state vector. The conditional expectation for the next period can be written using (4) as:

$$y_{t+1} = \hat{y}_{t+1} + u_{t+1} \tag{18}$$

where: $$\hat{y}_{t+1} = a + B\,\hat{x}_{t+1} + \sum\nolimits_{l=2}^{L} \beta_l' z_{t+2-l}$$

$$u_{t+1} = B\Gamma\upsilon_{t+1} + e_{t+1};\;\; B = [\beta_0 \beta_1'].$$

Since the macro errors are heteroscedastic in EA_1, so is u_t:

$$u_{t+1} \sim N\left(0_k, B\Gamma\left(\Delta_0 + \Delta_1 x_{1,t}\right)\Gamma' B'\right) \tag{19}$$

C. Yield Model Coefficients

The coefficients of (14) are derived in appendix 2. These coefficient systems are recursive in maturity. They are also recursive in the sense that $\Psi_{2,\tau}$ does not depend upon $\psi_{1,\tau-1}$ (or $\gamma_{\tau-1}$). This sub-structure is standard and common to both models, as is the recursion for the intercept term. The only difference between the EA_0 and EA_1 models is found in the two recursion relationships for the first slope coefficient. These are encompassed by the model:

$$\psi_{1,\tau} = \left(\xi^Q + j_1\right)\psi_{1,\tau-1} + \psi_{2,\tau-1}'\Phi_{21}^Q - \frac{1}{2}\psi_{2,\tau-1}'\Sigma_1\Phi_{2,\tau-1}$$

$$-\frac{1}{2}\delta_{11}\left(\psi_{1,\tau-1} + \psi_{2,\tau-1}'C_{21}\right)^2 \quad \tau = 1, ..., M. \tag{20}$$

where: $\Sigma_i = C_{22} D_i C_{22}';\;\; D_i = Diag\{\{\delta_{2i}, ..., \delta_{4i}\}, 0_{N-4}'\};\quad i = 0,1$

The EA_0 model simplifies this by setting δ_{11} (and hence Σ_1) to zero:

$$\psi_{1,\tau} = \left(\xi^Q + j_1\right)\psi_{1,\tau-1} + \psi_{2,\tau-1}'\Phi_{21}^Q;\quad \tau = 1, ..., 60. \tag{21}$$

where: $\xi^{Q} = \xi^{Q0} = \xi - \delta_{10}\lambda_{11}$; $\Phi_{21}^{Q} = \Phi_{21} - \Upsilon$. Υ is a 9 × 1 vector in which the first three elements are free parameters and the rest are zero. If $\xi^{Q_0} + j_1 = 1$, then (21) has a unit root and it can be shown that in the limit the forward rate falls without bound as shown in appendix 2, a problem originally pointed out by Campbell et al (1996).

The EA_1 model specifies the first coefficient of (20) as: $\xi^{Q} = \xi^{Q_1} = \xi - \delta_{11}\lambda_{10}$. It retains the quadratic terms, which show the Jensen effects implied by the square root volatility specification. As Campbell et al (1996) note, this means that $\psi_{1,\tau}$ and hence the forward rate have well defined asymptotes even if $\xi^{Q_1} + j_1 \geq 1$. This makes it more suitable for use with data sets such as the one used in this research that exhibit unit or near-unit roots.

The other slope coefficients are common to both models and follow the standard recursion:

$$\Psi_{2,\tau} = \left(\Phi_{22}^{Q}\right)' \Psi_{2,\tau-1} + J \tag{22}$$

$$= \left(I - \left(\Phi_{22}^{Q}\right)'\right)^{-1} \left(I - \left(\left(\Phi_{22}^{Q}\right)'\right)^{\tau}\right) J$$

where: $\Phi_{22}^{Q} = \Phi_{22} - \Lambda_{22}$. I assume that the roots of this system are stable under Q, so this has the asymptote:

$$\Psi_{2}^{*} = \lim_{\tau\to\infty} \Psi_{2,\tau} = \left(I - \left(\Phi_{22}^{Q}\right)'\right)^{-1} J. \tag{23}$$

The intercepts follow another standard recursion:

$$\Delta\gamma_{\tau} = \gamma_{\tau} - \gamma_{\tau-1} = \Psi'_{2,\tau-1}\Theta_{2}^{Q} + \psi_{1,\tau-1}\theta^{Q} - \frac{1}{2}\Psi'_{2,\tau-1}\Sigma_{0}\Psi_{2,\tau-1} - \frac{1}{2} \tag{24}$$

$$\delta_{10}\left[\psi_{1,\tau-1} + \psi'_{2,\tau-1}C_{21}\right]^{2}; \qquad \tau = 2, ..., M.$$

where: $\theta^{Q} = \theta - \delta_{10}\lambda_{10}$; $\Theta_{2}^{Q} = \Theta_{2} - F$;. In EA_1 there is a restriction across $\xi^{Q_1} = \xi - \delta_{11}\lambda_{10}$ and $\theta^{Q_1} = \theta - \delta_{10}\lambda_{10}$ because λ_{10} then determines both coefficients given ξ, θ, δ_{11}, δ_{10}. M1 enforces this restriction. The encompassing model M2 relaxes this restriction and defines θ^{Q} as a free parameter defined independently of the other parameters.

IV. Model Estimation and Evaluation

The macro (5) and yield (18) models are estimated jointly by maximum likelihood. Appendix 3 derives the likelihood function and describes the numerical optimisation procedure. Table 1 provides some basic summary statistics for these data. Preliminary work designed to estimate the dimensionality of the model estimated OLS regression equations for the inflation rate (π), the output gap (g); and the 3-month Treasury bill discount rate (r) using $y_{60,t}$ as a proxy for $x_{1,t}$. This system was estimated for L = 2, 3, 4 and 5 lags, with both homoscedastic and heteroscedastic error structures and the results suggested the use of a three-lag model. This gives a vector X_t of ten state variables (i.e. $x_{1,t}$ and current and two lagged values of z_t).

I began by estimating the standard macro-finance model $EA_0(10)$. This is homoscedastic and identifies the one period yield with the T-bill rate: $y_{1,t} = r_t$. With this dynamic structure it has 62 parameters[3]. The empirical version of this model is called M0 and has a loglikelihood value of 590.7 Model M1 is the empirical version of the equivalent $EA_1(10)$ specification. This uses another 4 parameters (for $\Delta_1(4)$) but saves one degree of freedom by using the restriction $\lambda_{11} = 0$. It has a loglikelihood value of 640.7. Model M2 relaxes this restriction and thus encompasses both M0 and M1. It employs a total of 66 parameters and has a loglikelihood value of 641.3 A standard likelihood ratio test of M0 against M2 gives a $\chi^2(4)$ value of 101.2, providing a decisive rejection of that model (the probability of observing this by chance is almost zero). However, M1 is acceptable ($\chi^2(1) = 1.2$; $p = 0.027$). Finally, I tested the standard macro-finance restriction: $y_{1,t} = r_t$ by treating the 10 coefficients of J as parameters to be estimated. This gave a significant increase in fit in all these models. However, only two of these

3. These are: θ; ξ; $\kappa(3)$; $\Phi_0(3)$; $\Phi_1(9)$; $\Phi_2(9)$; $\Phi_3(9)$; $G(3)$; $H(3)$; λ_{10}; λ_{11}; $\Lambda_{20}(3)$; $\Lambda_{21}(3)$; $\Lambda_{22}(9)$ and $\Delta_0(4)$.

TABLE 1. Data Summary Statistics: 1961Q4-2004Q1

	y_{60}	g	π	y_1	y_4	y_8	y_{12}	y_{20}	y_{28}	y_{40}
Mean	7.4383	0.0266	4.4345	5.8009	6.3954	6.6305	6.7849	7.0021	7.1478	7.2513
Std.	2.3833	2.3382	2.9841	2.7750	2.8174	2.7355	2.6508	2.5447	2.4790	2.4194
Skew.	1.8188	–0.575	1.4161	1.1907	1.8820	1.8697	1.8793	1.9633	1.9622	1.9215
Kurt.	0.3945	1.0230	1.5178	2.3986	1.1986	1.0622	0.9505	0.9166	0.7550	0.5929
Auto.	0.9971	0.4632	0.9921	0.9815	0.9892	0.9923	0.9944	0.9953	0.9963	0.9969
KPSS	0.4298	0.2151	0.3399	0.3100	0.3307	0.3348	0.3399	0.3475	0.3548	0.3761
ADF	–2.091	–4.133	–2.411	–2.110	–2.100	–2.063	–2.031	–2.043	–1.991	–1.951

Note: Inflation (π) and interest rate are from Datastream. Output gap (g) is from OECD. Yield data are US Treasury discount bond equivalent data compliled by McCulloch and Kwon (1990) updated by the New York Federal Reserve Bank. Mean denotes sample arithmetic mean expressed as percentage p.a.; Std. standard deviation and Auto. the first order quarterly autocorrelation coefficient. Skew. & Kurt. are standard measures of skewness (the third moment) and kurtosis (the fourth moment). KP SS is the Kwiatowski et al (1992) statistic testing the null hypothesis of level stationarity. The 10% and 5% significance levels are 0.347 and 0.463 respectively. ADF is the Adjusted Dickey-Fuller statistic testing the null hypothesis of non-stationarity. The 10% and 5% significance levels are 2.575 and 2.877 respectivly.

TABLE 2. Coefficients of determination (R^2), 1961Q4-2004Q1

	x_1	π	g	y_1	y_4	y_8	y_{12}	y_{20}	y_{28}	y_{40}
M0	0.967002	0.945406	0.930174	0.908774	0.947141	0.953988	0.963018	0.979067	0.986337	0.993420
M1	0.966918	0.945504	0.930304	0.910334	0.946610	0.954304	0.962788	0.979295	0.986673	0.993513
M3	0.966953	0.945865	0.931198	0.909153	0.949798	0.956978	0.963845	0.979441	0.986798	0.993490

TABLE 3. Dynamic model structures (asymptotic t-values in parentheses.)

Parameter	M0	M3	Parameter	M0	M3	Parameter	M0	M3
Φ_0			κ			θ	0.00019	0.00019
							(1.27)	(1.05)
$\varphi_{0,1}$	0.09893	0.09349	κ_1	–0.00080	0.00012	ξ	0.99123	0.99030
	(4.03)	(4.03)		(1.42)	(1.42)		(52.11)	(50.00)
$\varphi_{0,2}$	0.04289	0.04325	κ_2	0.00131	0.00151	j_1	0	0.11730
	(3.13)	(3.33)		(7.00)	(6.69)		(–)	(16.11)
$\varphi_{0,3}$	0.03978	0.039753	κ_3	0.00016	0.00010	j_r	1	0.91912
	(6.79)	(7.00)		(7.03)	(7.44)		(–)	(52.11)
Φ_1			Φ_2			Φ_2		
$\varphi_{1,11}$	1.15284	1.15183	$\varphi_{2,11}$	–0.12278	–0.12549	$\varphi_{3,11}$	–0.087651	–0.08763
	(16.10)	(16.88)		(0.90)	(0.86)		(1.12)	(1.08)
$\varphi_{1,21}$	0.07510	0.07800	$\varphi_{2,21}$	0.01079	0.014413	$\varphi_{3,21}$	0.00695	0.00977
	(3.17)	(3.00)		(0.36)	(0.06)		(1.67)	(1.84)
$\varphi_{1,31}$	0.12002	0.115967	$\varphi_{2,31}$	–0.11334	–0.08392	$\varphi_{3,31}$	–0.03726	–0.03726
	(0.40)	(0.30)		(1.67)	(1.90)		(1.48)	(1.21)
$\varphi_{1,12}$	–0.13091	–0.13065	$\varphi_{2,12}$	–0.00231	–0.00198	$\varphi_{3,12}$	0.03830	0.03860
	(1.96)	(1.75)		(0.69)	(0.60)		(0.57)	(0.37)
$\varphi_{1,22}$	1.06812	1.06836	$\varphi_{2,22}$	0.05350	0.5400	$\varphi_{3,22}$	–0.17656	–0.17597
	(20.36)	(23.83)		(2.03)	(2.44)		(8.89)	(8.90)
$\varphi_{1,32}$	0.02698	0.02734	$\varphi_{2,32}$	–0.25956	–0.27996	$\varphi_{3,32}$	0.17881	0.17846
	(1.03)	(1.09)		(0.17)	(0.09)		(2.38)	(2.40)

(*Continued*)

TABLE 3. (Continued)

Parameter	M0	M3	Parameter	M0	M3	Parameter	M0	M3
$\varphi_{1,13}$	0.04144	0.04148	$\varphi_{2,13}$	0.11037	0.11052	$\varphi_{2,13}$	–0.09688	–0.09673
	(2.00)	(2.08)		(1.96)	(1.90)		(0.60)	(0.61)
$\varphi_{1,23}$	0.22131	0.22143	$\varphi_{2,23}$	0.05114	0.05138	$\varphi_{2,23}$	–0.21724	–0.21697
	(1.00)	(1.05)		(4.04)	(4.91)		(14.89)	(16.02)
$\varphi_{1,33}$	0.96971	0.97046	$\varphi_{2,33}$	–0.29392	–0.29257	$\varphi_{2,33}$	0.25036	0.25038
	(7.72)	(7.58)		(16.83)	(16.99)		(1.93)	(2.06)

parameters were statistically significant, those attached to the financial factor (J_{x1}) and the T-bill rate (J_r). Adding these to model M1 gives my preferred specification M3. This has 67 parameters and a loglikelihood of 648.3, revealing a significant improvement upon M1 ($\chi^2(2) = 15.0$; p $\simeq$ 0).

These tests strongly support the stochastic volatility hypothesis: introducing the 4 conditioning parameters of Δ_1 dramatically increases the likelihood. This parallels the results of Duffee (2002) and others using the conventional yield-factor model. This modification has two effects (a) it introduces quadratic Jensen terms into the yield coefficient associated with the financial factor (20) and (b) it allows for conditional volatility in the data. Theoretically, these two effects are inextricably linked because the yield structure depends upon the stochastic structure. However, it is possible to analyze themseparately. Table 2 reports the R^2 statistics associated with the 10 equations of M1, M1 and M3, showing that these models have similar prediction errors despite the questionable mathematical properties of the M0 yield specification. These are only marginally higher in M3 than in M0. This is perhaps not surprising since these models are all linear in variables and the macro-dynamic structures are identical. Their estimated coefficients, (both the macro parameters and the factor loadings) are numerically very similar. In fact, the main reason why the likelihood of the EA_1 models are so much higher is because of effect (b) they allow for conditional volatility in the data, damping the effect that large residuals have on the likelihood value as explained in the next section. This result again parallels that of Duffee (2002) for the conventional model. In other words, the information that the estimation procedure uses to pin down the Δ_1 parameters comes largely from the heteroscedastic behavior of the data rather than the behavior of the yield curve itself.

I now look at the empirical results in detail. The parameters of M0 and M3 are set out in Tables 3, 4 and 5. These are generally well determined, although as we would expect in VAR-type analysis, some of the off-diagonal dynamic coefficients are insignificant. As in previous studies of essentially affine yield structures, many of the risk-adjustment parameters are poorly determined. Since the macro parameters and factor loadings for the alternative specifications are similar, I focus on the EA_1 model and in particular the insights it yields into the stochastic volatility structure.

TABLE 4. Variance structures (asymptotic t-values in parentheses)

Parameter	M0	M1	Parameter	M0	M1
Δ_0			Δ_1		
δ_{10}	1.56567×10^{-5}	6.9077×10^{-6}	δ_{11}		3.1310×10^{-7}
	(8.04)	(0.99)		(–)	(2.75)
δ_{20}	1.70580×10^{-6}	2.5990×10^{-4}	δ_{21}		2.3277×10^{-6}
	(5.07)	(1.99)		(–)	(3.17)
δ_{30}	3.1768×10^{-5}	8.9370×10^{-5}	δ_{31}		3.5395×10^{-6}
	(5.94)	(2.77)		(–)	(3.33)
δ_{40}	4.5083×10^{-6}	1.4825×10^{-3}	δ_{41}		7.0130×10^{-6}
	(4.67)	(1.21)		(–)	(4.9)
G			H		
g_{21}	0.18823	0.13631	h_1	–0.06003	–0.04001
	(6.21)	(1.88)		(0.21)	(0.12)
g_{31}	0.33294	0.23055	h_2	0.16733	0.13103
	(10.12)	(3.83)		(10.12)	(4.35)
g_{32}	0.14805	0.05401	h_3	–0.16462	–0.24268
	(9.12)	(8.08)		(4.12)	(5.93)

A. The Stochastic Structure

At the core of this model there is an autoregressive system (5) determining the macro-dynamics. The novelty here is the introduction of square root volatility effects into this structure. The time variation in volatility is driven by the financial factor $x_{1,t}$ inferred from (16):

$$x_{1,t} = \underset{(6.02)}{-0.005465} + y_{60,t} \underset{(0.51)}{-0.01869\pi_t} \underset{(2.38)}{-0.09208 g_t} \underset{(4.55)}{-0.20833 r_t}$$

$$\underset{(0.36)}{-0.00237\pi_{t-1}} \underset{(1.64)}{+0.04486 g_{t-1}} \underset{(1.98)}{+0.01857 r_{t-1}} \underset{(0.55)}{+0.01814\pi_{t-2}}$$

$$\underset{(1.42)}{+0.06098 g_{t-2}} \underset{(3.33)}{-0.06762 r_{t-2}}$$

This factor is dominated by the current value of the long bond yield,[4]

4. The models of risk premia developed by Glosten et al (1993) and Scruggs (1998) are similar in this respect. The first conditions volatility on the yield gap and the second on a

TABLE 5. Risk adjustment structures (asymptotic t-values in parentheses.)

Parameter	M0	M3	Parameter	M0	M3	Parameter	M0	M3
Λ			F			Y		
λ_{10}	–39.9	–42.344	F_1	–1150.23	–111.923	υ_1	–0.11770	–0.10070
	(3.93)	(5.83)		(13.23)	(11.23)		(8.42)	(8.42)
λ_{11}	–13.54	(–)	F_2	–241.6.933	–160.933	υ_2	–0.02034	–0.02246
	(1.30)			(8.87)	(7.00)		(1.92)	(1.92)
			F_3	1.256	39.656	υ_3	–0.0432	–0.0434
				(1.77)	(2.77)		(2.21)	(2.21)
Λ_{22}								
$\lambda_{22,11}$	0.03162	0.03621	$\lambda_{22,13}$	–0.09051	–0.09038	$\lambda_{22,12}$	–0.19075	–0.17065
	(0.41)	(0.41)		(1.14)	(1.14)		(0.18)	(0.18)
$\lambda_{22,21}$	0.03500	0.03480	$\lambda_{22,23}$	–0.16844	–0.16837	$\lambda_{22,22}$	0.07407	0.06797
	(2.41)	(2.41)		(1.19)	(1.19)		(0.10)	(0.10)
$\lambda_{22,31}$	–0.01092	–0.01097	$\lambda_{22,33}$	0.01307	0.01311	$\lambda_{22,32}$	–0.04176	–0.04173
	(1.26)	(1.26)		(1.10)	(1.10)		(0.10)	(0.10)

short term interest rate.

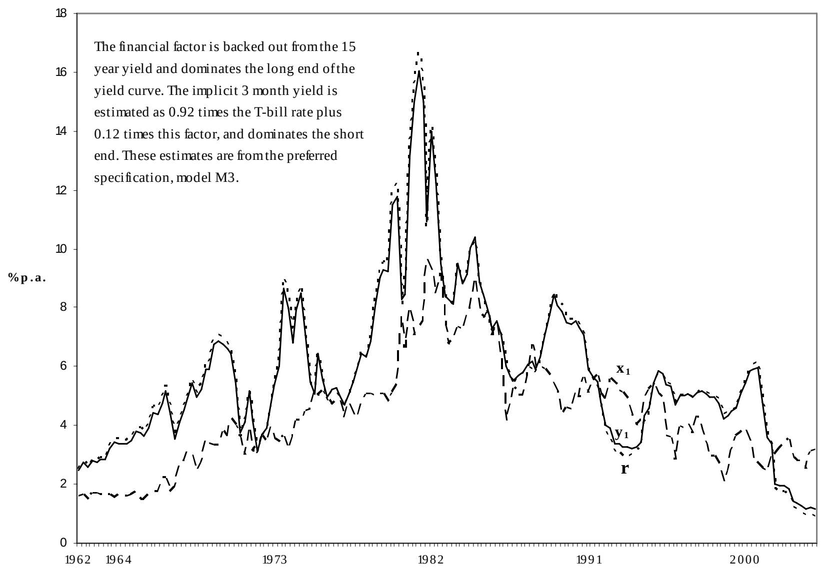


FIGURE 3.— Policy rate (r), one period yield (y_1) and financial factor (x_1)

which dominates the behavior of the long end of the yield curve. The short end of the curve is dominated by the implicit 3 month yield which is estimated as $y_{1,t} = 0.92r_t + 0.12x_{1,t}$. The factor loadings are reported in figure 7. This shows how the loading on the T-bill rate (r_t) decays and that on the financial factor ($x_{1,t}$) increases with maturity in M3. These three rates are depicted in figure 3.

The dynamic properties of the model are dominated by the autoregressive coefficient associated with $x_{1,t}$, which is close to unity under both measures. Solving the model conditional upon $x_{1,t}$ shows the steady state effect of a permanent percentage point increase in $x_{1,t}$ would be to raise the steady state rate of inflation by 0.448, the T-bill rate by 0.822 and the 15 year rate by 1.038 points, implying a rise in both the real rate of interest and the risk premium. Consequently it seems to reflect expectations about both underlying inflation and real interest rates.

The M3 model estimates of the financial factor are shown in Figure 4(a). This shows the one-period ahead expectation along with the 95% confidence interval computed from the one-period ahead conditional

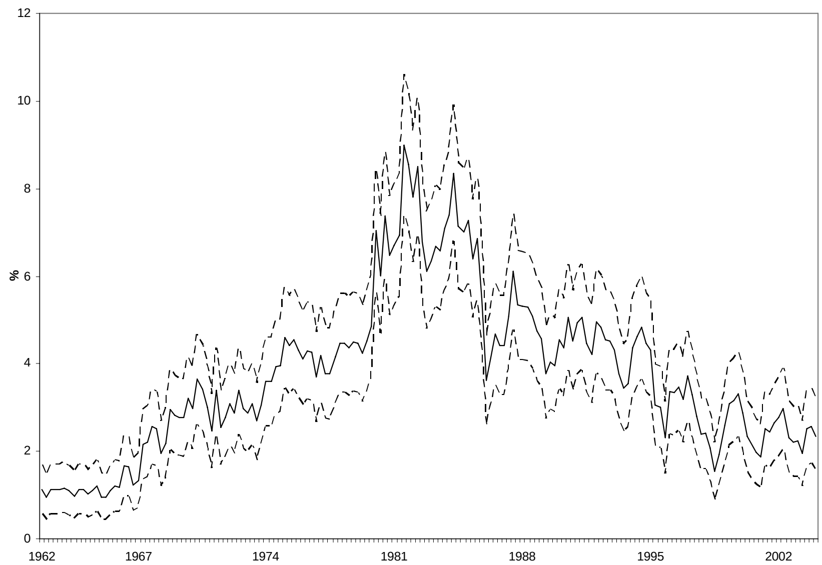


FIGURE 4(a).— Variability of the financial factor (One step ahead estimate and 95% confidence interval)

volatility.[5] This interval rises with $x_{1,t}$ during the 1970s and after that both subside. This factor also drives the volatility in the other model variables. The one-quarter-ahead forecast values and 95% confidence intervals for the three macro variables and the 5 and 10 year yields are shown in figures 4(b)-(e). The effect of stochastic volatility is particularly pronounced in the case of the T-bill rate, consistent with the finding in univariate models (Chen and Scott (1993), Ait-Sahalia (1996), Stanton (1997) and others). Its variance is low over the first four years and last two years of the estimation period, consistent with the ex post stability of interest rates over these periods (figure 4(d)). These fluctuations in volatility are a very important factor in explaining the superior performance of the EA_1 models. That is because the likelihood function (42) normalizes the squared prediction errors in the sum of squares by the conditional variances, as in E((22) of Duffee (2002). In M1-M3 these variances depend upon $x_{1,t}$ and this helps to reduce the

5. These intervals are computed as 1.96 times the standard deviations implied by the square roots of the conditional variances (9) and (19).

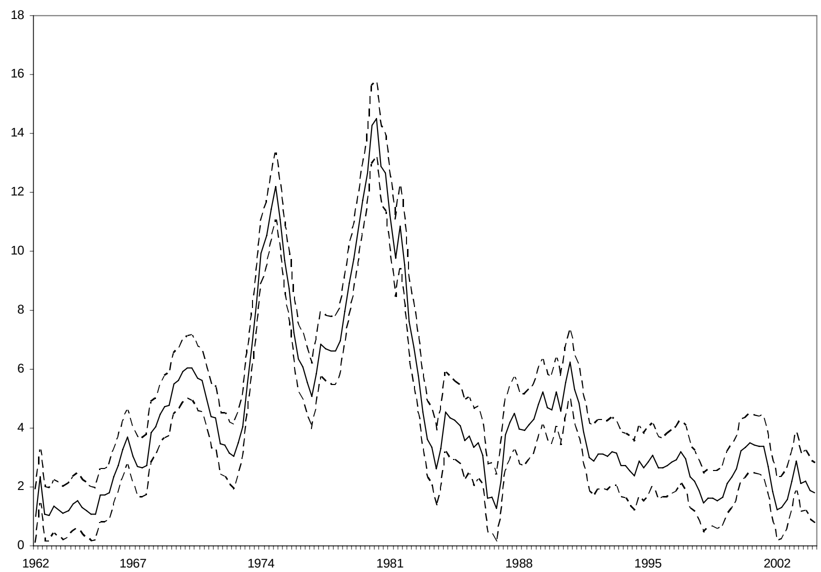


FIGURE 4(b).— Inflation variability (One step ahead estimate and 95% confidence interval)

impact of the large errors that tend to occur when this is high. Consequently the likelihood is much higher than for the constant variance model M0, even though the un-normalized R^2 and RMSEs of the macro and yield forecasts of these models are similar.

This effect can be seen using a simple two-stage OLS method. That is because the conditional means and variances are linear in the financial factor, which can be approximated by the (lagged) 15 year yield r_{60}. Using this to first explain its own conditional mean, we run the first stage OLS regression (using my 1961-2004 data set):

$$y_{60,t+1} = \underset{(1.68)}{6.11\times 10^{-4}} + \underset{(51.98)}{0.9678}\, y_{60,t} + w_{1,t+1}$$

We then use $y_{60,t}$ to explain the conditional variance, represented by the squared first-stage residuals. This gives an approximation to the volatility equation (7):

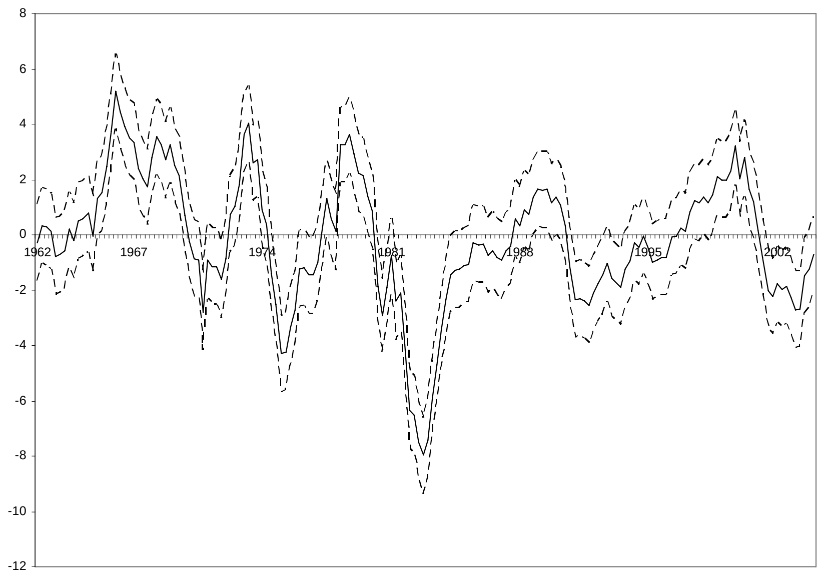


FIGURE 4(c).— Output gap variability (One step ahead estimate and 95% confidence interval)

$$w_{1,t+1}^2 = \underset{(3.68)}{-1.61534\times 10^{-3}} + \underset{(5.86)}{0.13142}\, y_{60,t}$$

The slope coefficient in this regression suggests that conditional volatility is very significant statistically.

B. The Dynamic Structure

How firmly does the financial factor anchor inflation and interest rates? This question depends upon whether they are co-integrated with the non-stationary nominal factor $x_{1,t}$. This was checked by running ADF tests on the residuals of these equations, which decisively reject nonstationarity. The macro variables adjust surprisingly quickly and smoothly to their equilibrium values (conditional upon $x_{1,t}$). This is clear from the impulse responses, which show the dynamic effects of innovations in the macroeconomic variables on the system. Because these innovations are correlated empirically, I use the orthogonalized

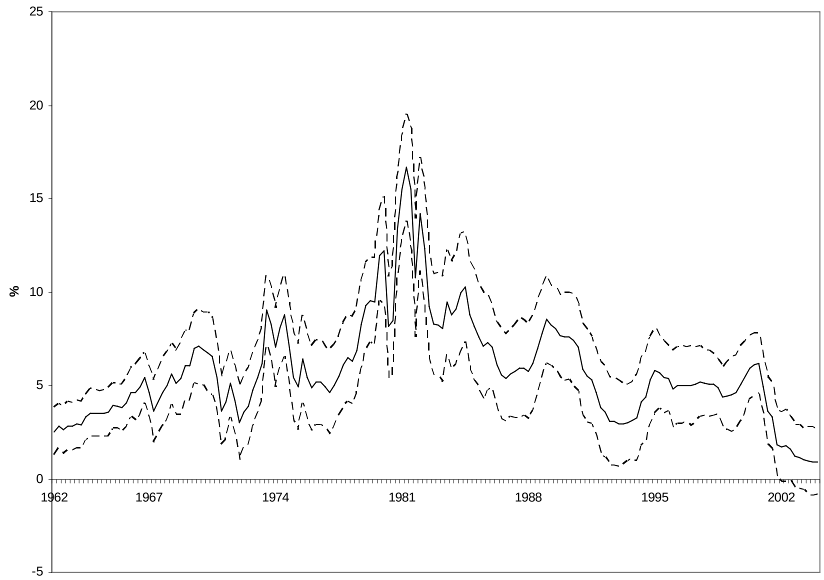


FIGURE 4(d).— Variability of T-bill rate (One step ahead estimate and 95% confidence interval)

innovations obtained from the triangular factorization defined in (4). The impulse responses show the effect on the macroeconomic system of increasing each of these shocks by one percentage point for just one period using the Wold representation of the system as described for example in Cochrane (1997).

This arrangement is affected by the ordering of the macro variables in the vector x_t, making it sensible to order the variables in terms of their likely degree of exogeneity or sensitivity to contemporaneous shocks. The financial factor is assumed to represent exogenous expectational influences, so this is ordered first in the sequence. This means that independent shocks to inflation, output and interest rates can then be interpreted as sudden shocks that are not anticipated by the bond market. Following Hamilton (1994) inflation is ordered before the output gap, on the view that macroeconomic shocks are accommodated initially by output rather than price. Interest rates are placed after these variables on the view that monetary policy reacts relatively quickly to disturbances in output and prices. Thus the variable ordering is: $x_{1,t}$; π_t; g_t and r_t. This means that shocks to the financial factor (v_1) disturb all

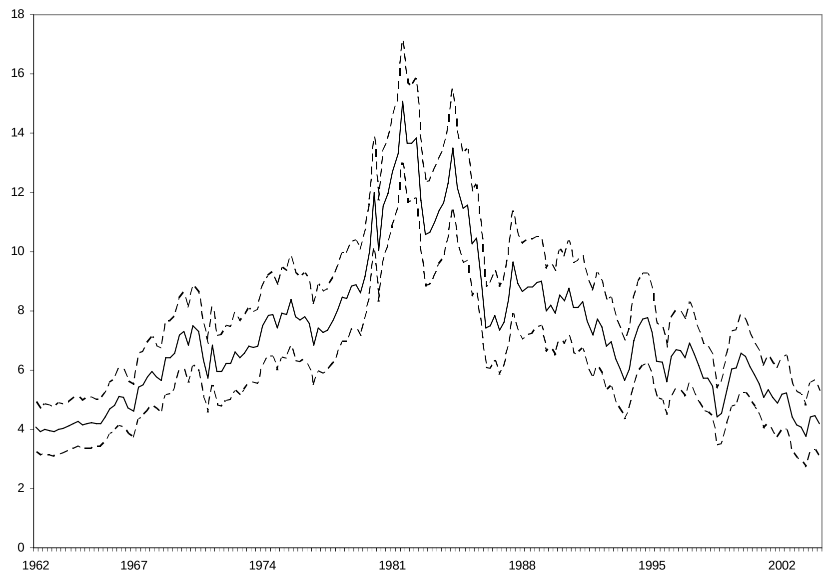
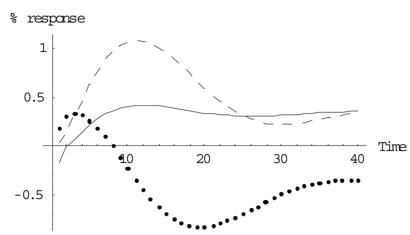


FIGURE 4(e).— Variability of 10 year yield (One step ahead estimate and 95% confidence interval)

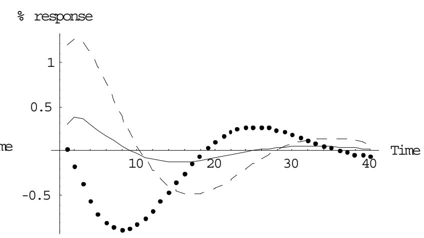
four variables contemporaneously as indicated by the first column of the matrix Γ shown in (4), independent shocks to inflation (v_2) affect output and interest rates but not the financial factor, and so on.

Figure 5 shows the results of this exercise. The continuous line shows the effect of each independent shock on the T-bill rate, the broken line the effect on inflation and the dotted line the effect on output. Elapsed time is measured in quarters. Panel (i) shows the effect of a shock to the financial factor (v_1). This could reflect an increase in the bond market's expected rate of inflation or the underlying real rate of return in the economy. Output and the T-bill rate increase immediately, but inflation does not, meaning that real interest rates increase initially. The financial factor acts as a leading indicator for inflation, which peaks after three years.

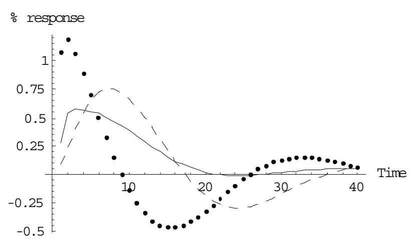
Panel (ii) shows the effect of an independent shock to inflation (v_2), essentially an inflationary impulse that is not anticipated by the bond market. The initial effect on the T-bill rate is only about a quarter of a point, so real interest rates fall. However, output falls back, reaching a trough after falling by 0.8% after two years, reflecting real balance and other contractionary inflationary effects. The fall in output has the effect of reversing the rise in inflation, setting up cycles in these variables. In



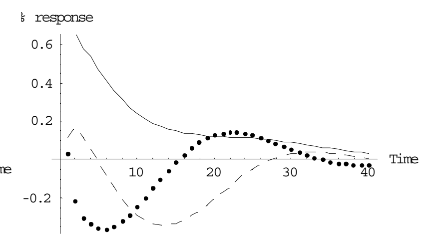
(i) Financial factor (f)



(ii) Inflation (π)



(iii) Output gap (g)



(iv) T-bill rate (r_1)

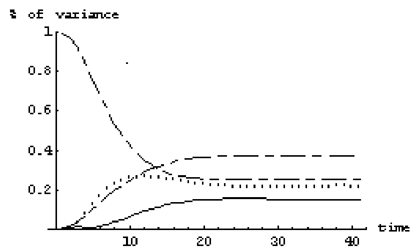
Key - effects on:
– – – – inflation •••••• output ——— T-bill rate

FIGURE 5.— Model M3 macroeconomic impulse responses

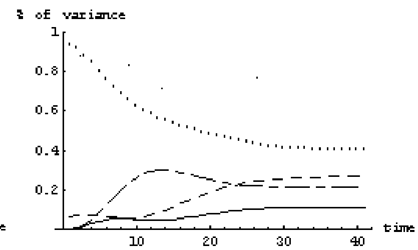
Note: Each panel shows the effect of a shock to one the four orthogonal innovations (υ_t) shown in (4). These shocks increase each of the factors in turn by one percentage point compared to its historical value for just one period. Since $x_{1,t}$ has a near-unit root, the first shock ($\upsilon_{1,t}$) has a persistent effect, while other shocks are transient. The continous line shows the effect on the spot rate, the dashed line the effect on output and the dotted line the effect on inflation. Elapsed time is measured in quarters.

contrast to the effects shown in the first panel, which are highly persistent, the system is close to its initial level after 10 years following this inflationary impulse. The other two panels show similarly fast responses, with qualitative effects in accordance with macroeconomic theory.

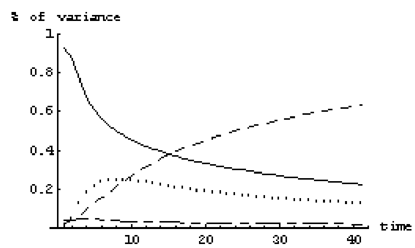
These responses are reflected in figure 6, which report the results of an Analysis of Variance (ANOVA) exercise. These figures show the share of the total variance attributable to the innovations at different lag lengths and are also obtained using the Wold representation of the system as described in Cochrane (1997). They indicate the contribution



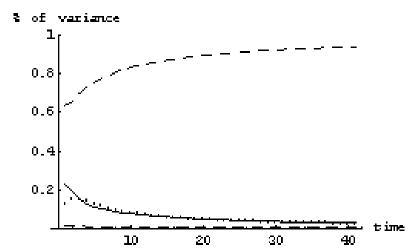
(i) Inflation variance



(ii) Output variance



(iii) Variance of T-bill rate



(iv) Variance of 5 year yield

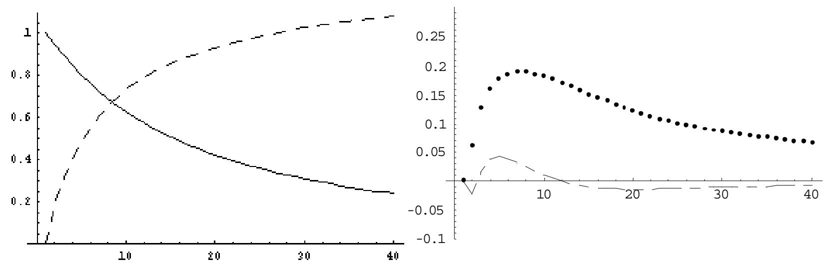
Key - % of variance due to orthogonal innovations in:
– – – – financial factor; – — – inflation; ········ output; —— T-bill rate.

FIGURE 6.— Model M3 Analysis of Variance

Note: Each panel shows the contribution to total variance of innovations in the orthogonal shocks representing innovations in each of the four driving variables. Elapsed time is measured in quarters.

each innovation would make to the volatility of each model variable if the error process was started in the first period. Initially, the variances of these variables are strongly influenced by their own innovations. However the influence of the long bond innovations builds up over time, particularly in the case of the T-bill rate, where this explains over half of the total forecast variance after 10 years.

Figure 7 shows the factor loadings as a function of maturity expressed in quarters. The first panel shows the loadings on r_t (continuous line) and $x_{1,t}$ (broken line). The spot rate is the link between the macro model and the term structure. Recall that in M3 it is estimated as $y_{1,t} = 0.92r_t + 0.12x_{1,t}$. These regression weights determine the first quarter loadings on r_t and $x_{1,t}$, while other factors



Financial factor, $x_{1,t}$ (– – –) and T-bill rate r_t (——) Output g_t (·····) and inflation π_t (— – —)

FIGURE 7.— Model M3 Factor loadings

Note: The factor loadings show the cumulative effect (after three quarters) of changes in the four factors on yields at different maturities

have a zero loading. The loadings on r_t then tend to decline monotonically with maturity, reflecting the relatively fast adjustment process. This mean it acts like the 'slope' factor in the conventional 3-factor model. In contrast, the slow-moving nature of $x_{1,t}$ means that its loading increases with maturity over most of this range, allowing it to act as a 'level' factor. The next panel shows the loadings on π (dotted line) and g (broken line).

The lower right hand panel of Figure 6 decomposes the conditional forecast variance of the 5 year yield into the separate effects of surprises to the four orthogonal shocks defined in (4). (ANOVA figures for the 7 and 10 year yields show a similar pattern.) Innovations in the three macro have a modest contribution for near-term forecasts, but are increasingly dominated by innovations in the long bond innovations. This explains over 95% of the total forecast variance 10 years ahead.

V. Conclusion

Conditional volatility is a common feature of macroeconomic and financial data and as Duffee (2002) and many others have shown, it is important to allow for this when modelling the yield curve and derivatives that are priced off this. My specification extends the new macro-finance model to allow for conditional volatility, bringing it into line with the conventional finance model. It is an EA_1 specification that conditions the central tendency and the variance structure of the model on the financial factor, which is closely correlated with the long bond yield. The likelihood of the new model is much higher than that of the existing EA_0 macro-finance specification, even though the raw forecast errors of the two models are similar. As was found to be the case in the conventional yield factor model (Duffee (2002)) this is because the EA_1 model allows for conditional volatility in the factors driving the system, damping the negative effect that large residuals have on the likelihood value. In practice then, the information that the estimation procedure uses to pin down the parameters of my EA_1 yield model comes indirectly from the behavior of the macroeconomic and latent factors rather than the behavior of the yield curve itself.

My model can be seen as a modification of the conventional EA_1 yield factor model of the bond market which replaces some of the latent factors by macroeconomic variables. It shows that the stochastic volatility identified by the conventional model is related to macroeconomic volatility. It can use a third-order dynamic specification with large number state variables (10) in place of the conventional first-order system because its parameters are estimated using by macroeconomic as well as yield data. However, the behavior of the yield curve is largely dictated by three factors: the financial factor, the output gap and the T-bill rate. The model is consistent with the traditional three latent factor US finance specification in this respect, but aligns the last two factors with observable variables. This research opens the way to a much richer term structure specification, incorporating the best features of the macro-finance and conventional finance models.

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Prof. P. Theodossiou, Editor-in-Chief, September 2008

Appendix 1: The State-Space Representation of the Model

Stacking (4) puts the system into state space form (5), where :

$$\Theta' = \left\{\theta, \kappa, 0_{N-4,1}\right\}; \tag{25}$$

$$\Phi = \begin{bmatrix} \xi & 0_{1,3} & \dots & 0_{1,3} & 0_{1,3} \\ \Phi_0 & \Phi_1 & \dots & \Phi_{l-1} & \Phi_l \\ 0_{3,1} & I_3 & .. & 0_{3^2} & 0_{3^2} \\ & & & & \\ 0_{3,1} & 0_{3^2} & \dots & I_3 & 0_{3^2} \end{bmatrix} = \begin{bmatrix} \xi & 0'_{N-1} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}.$$

and where the last matrix partitions Φ conformably with (6), so that Φ_{21} is $(N-1) \times 1$ and Φ_{22} is $(N-1)^2$. Similarly:

$$C = \begin{bmatrix} 1 & 0_{1,3} & 0_{1,6} \\ H & G & 0_{3,6} \\ 0_{6,1} & 0_{6,3} & 0_{6^2} \end{bmatrix} = \begin{bmatrix} 1 & 0'_{N-1} \\ C_{21} & C_{22} \end{bmatrix}. \tag{26}$$

where the last matrix partitions C conformably with (6): C_{21} is $(N-1) \times 1$ and C_{22} is $(N-1)^2$. The error structure of (6) follows from (8) as:

$$W_{2,t+1} = C_{21} w_{1,t+1} + C_{22} S_t U_{2,t+1} \tag{27}$$

$$U_{2,t+1} \sim N\left(0_{N-1}, I_{N-1,3}\right)$$

where: $I_{N-1,3} = Diag\{1_3, 0_{N-4}\}$ and $S_t = Diag\left\{\left\{\left(\delta_{20} + \delta_{21} x_{1,t}\right)^{\frac{1}{2}}, \dots, \left(\delta_{40} + \delta_{41} x_{1,t}\right)^{\frac{1}{2}}\right\}, 0'_{N-4}\right\}$; $U'_{2,t+1} = \left\{u'_{2,t+1}, 0'_{N-4}\right\}$. This implies the conditional expectations:

$$E_t\left\{\exp\left[\Psi'_2 U_{2,t+1}\right]\right\} = \exp\left\{\frac{1}{2}\left[\Psi'_2 I_{N-1,3} \Psi_2\right]\right\}; \tag{28}$$

$$E_t\left\{\exp\left[\Psi'_2 S_t U_{2,t+1}\right]\right\} = \exp\left\{\frac{1}{2}\left[\Psi'_2 S_t I_{N-1,3} S_t \Psi_2\right\} = \exp\left\{\frac{1}{2}\left[\Psi'_2 S_t^2 \Psi_2\right]\right\}.$$

where:

$$S_t^2 = D_0 + x_{1,t} D_1 \tag{29}$$

where: $D_i = Diag\{\{\delta_{2i}, ..., \delta_{4i}\}, 0'_{N-4}\}; \quad i = 0,1.$

Appendix 2 : The EA_0 and EA_1 Specifications

This appendix derives the arbitrage-free bond price systems for the EA_0 and EA_1 specifications. Substituting (6), (34) and (27) into (11), noting that $w_{1,t+1}$ and $U_{2,t+1}$ are independent allows this to be factorized as:

$$\begin{aligned} P_{\tau,t} &= E_t\left[M_{t+1}P_{\tau-1,t+1}\right] \\ &= \exp\left\{-\left[\omega_t + y_{1,t} + \gamma_{\tau-1} + \psi_{1,\tau-1}\left(\theta + \xi x_{1,t}\right) + \Psi'_{2,\tau-1}\left(\Theta_2 + \Phi_{21}x_{1,t} + \right.\right.\right. \\ &\left.\left.\left. \Phi_{22}X_{2,t}\right)\right]\right\} \times E_t\left\{\exp\left[\left(\Psi'_{2,\tau-1}C_{22}S_t + \Lambda_{2,t}\right)' U_{2,t+1}\right]\right\} \\ &\times E\left\{\exp\left[\left(\psi_{1,\tau-1} + \lambda_{1,t} + \Psi'_{2,\tau-1}C_{21}\right) w_{1,t+1} \middle| x_{1,t}\right\}\right. \end{aligned} \tag{30}$$

These errors are all Gaussian and are evaluated using (7) and (28):

$$\begin{aligned} P_{\tau,t} = \exp\Big\{&-\left[\omega_t + y_{1,t} + \gamma_{\tau-1} + \psi_{1,\tau-1}\left(\theta + \xi x_{1,t}\right) + \Psi_{2,\tau-1}\left(\Theta_2 + \Phi_{21}x_{1,t} + \right.\right. \\ &\left.\left. \Phi_{22}X_{2,t}\right)\right\} + \frac{1}{2}\left(\Psi'_{2,\tau-1}C_{22}S_t + \Lambda'_{2,t}\right) I_{N-1,3}\left(S_t C'_{22}\Psi_{2,\tau-1} + \Lambda_{2,t}\right) \\ &+ \frac{1}{2}\left(\delta_{10} + \delta_{11}x_{1,t}\right)\left(\psi_{1,\tau-1} + \lambda_{1,t} + \Psi'_{2,\tau-1}C_{21}\right)^2\Big]\Big\} \end{aligned} \tag{31}$$

In the case of a one period bond $P_{1,t} = \exp\{-y_{1,t}\}$, which gives the initial conditions:

$$\gamma_1 = 0; \psi_{1,1} = j_1; \Psi_{2,1} = J'_2 \tag{32}$$

and the restriction:

$$\omega_t = \frac{1}{2}\left(\Lambda'_{2,t} I_{N-1,3} \Lambda_{2,t} + \left(\delta_{10} + \delta_{11} x_{1,t}\right) \lambda^2_{1,t}\right);$$

Substituting this and (29) into (31):

$$P_{\tau,t} = \exp\left\{-\left\{\gamma_{\tau-1} + y_{1,t} + \psi_{1,\tau-1}\left(\theta + \xi x_{1,t}\right) + \Psi_{2,\tau-1}\left(\Theta_2 + \Phi_{21} x_{1,t} + \Phi_{22} X_{2,t}\right)\right\} + \Psi'_{2,\tau-1} C_{22} S_t \Lambda_{2,t} + \left(\delta_{10} + \delta_{11} x_{1,t}\right)\left(\psi_{1,\tau-1} + \Psi'_{2,\tau-1} C_{21}\right)\lambda_{1,t} + \frac{1}{2}\Psi'_{2,\tau-1}\left(\Sigma_0 + \Sigma_1 x_{1,t}\right)\Psi_{2,\tau-1} + \frac{1}{2}\left(\delta_{10} + \delta_{11} x_{1,t}\right)\left(\psi_{1,\tau-1} + \Psi'_{2,\tau-1} C_{21}\right)^2\right\} \quad (33)$$

The price parameter systems are obtained by specifying the prices of risk parameters. Following Duffee (2002) I define the (N – 1) × 1 deficient vector $\Lambda_{2,t} = \left[\lambda'_{2,t}, 0'_{N-4}\right]'$ and write the log SDF as:

$$-m_{t+1} = \omega_t + y_{1,t} + \lambda_{1,t} w_{1,t+1} + \Lambda'_{2,t} U_{2,t+1} \quad (34)$$

and then assume:

$$\lambda_{1,t} = \lambda_{10} + \lambda_{11} x_{1,t} + \Lambda_{12} \mathrm{X}_{2,t} \quad (35)$$

$$\Lambda_{2,t} = S_t C'_{22} \Lambda_{20} + S_t^{-1} C'^{-1}_{22} \Lambda_{21} x_{1,t} + S_t^{-1} C'^{-1}_{22} \Lambda_{22} X_{2,t}$$

where: Λ_{20} and Λ_{21} are $(N – 1) \times 1$ and Λ_{22} is $(N – 1)^2$. The elements of the last $N – n$ rows of these matrices (and the last $N – n$ columns of Λ_{22}) are zero. To obtain an affine yield solution for the EA_1 model it is necessary to assume: $\lambda_{11} = 0$; $\Lambda'_{12} = 0_{N-1}$. To allow the EA_1 model to encompass EA_0 I also assume that: $\Lambda'_{12} = 0_{N-1}$ for EA_0, checking that this was acceptable at the 95% significance level. Substituting these formulae into (33):

$$P_{\tau,t} = \exp\Big\{-\Big\{x_{1,t}\Big[\psi_{1,\tau-1}\big(\xi + j_1 - \delta_{11}\big(\lambda_{10} + \lambda_{11}x_{1,t}\big) + \Psi'_{2,\tau-1}\big(\Phi_{21} - \Sigma_1\Lambda_{20}$$

$$-\Lambda_{21} - \delta_{11}\lambda_{10}C_{21}\big) - \frac{1}{2}\Psi'_{2,\tau-1}\Sigma_1\Psi_{2,\tau-1} - \frac{1}{2}\delta_{11}\big(\psi_{1,\tau-1} + \Psi'_{2,\tau-1}C_{21}\big)^2\Big]$$

$$+\big[J'_2 + \Psi'_{2,\tau-1}\big(\Phi_{22} - \Lambda_{22}\big)\big]X_{2,t} + \gamma_{\tau-1} + \psi_{1,\tau-1}\big(\theta - \delta_{10}\lambda_{10}\big)$$

$$+\Psi'_{2,\tau-1}\big(\Theta_2 - \Sigma_0\Lambda_{20} - \delta_{10}\lambda_{10}C_{21}\big) - \frac{1}{2}\Psi'_{2,\tau-1}\Sigma_0\Psi_{2,\tau-1} - \frac{1}{2}\delta_{10}$$

$$\big(\psi_{1,\tau-1} + \Psi'_{2,\tau-1}C_{21}\big)^2\Big\}\Big\}. \qquad (36)$$

For the EA_1 models M1-M3, (20), (22) and (24) follow by setting $\lambda_{1,t} = \lambda_{10}$ and equating the coefficients of X_t in the exponent with those in (14). In this case Υ is interpreted in terms of the risk parameters as $\Upsilon = \left(\Sigma_1\Lambda_{20} + \Lambda_{21} + \delta_{11}\lambda_{10}C_{21}\right)$. Setting δ_{11} and Σ_1 to zero and equating coefficients gives the parameter systems for the EA_0 model, including (21). In this case: $\Upsilon = \Lambda_{21} + \delta_{10}\lambda_{11}C_{21}$.(22) and (24) are shared with EA_1, where $F = \Sigma_0\Lambda_{20} - \delta_{10}\lambda_{10}C_{21}$

A. Forward Rates and Risk Premia

The τ – period ahead forward interest rate is defined as $f_{\tau,t} = p_{\tau,t} - p_{\tau+1,t}$:

$$f_{\tau,t} = \gamma_{\tau+1} - \gamma_\tau + \big(\psi_{1,\tau+1} - \psi_{1,\tau}\big)x_{1,t} + \big[\Psi_{2,\tau+1} - \Psi_{2,\tau}\big]' X_{2,t}; \quad \tau = 1, ..., M \qquad (37)$$

The risk premia follow by setting the price of risk parameters to zero in (36). This is equivalent to setting M_{t+1} in (30) to $\exp\{-y_{1,t}\}$ and gives the discounted expectation under *P*:

$$\exp\big[-y_{1,t}\big]E_t\big[P_{\tau-1,t+1}\big] = \exp\big\{-\big[\gamma_{\tau-1} + y_{1,t} + \psi_{1,\tau-1}\big(\theta + \xi x_{1,t}\big) + \Psi_{2,\tau-1}$$

$$\left(\Theta_2 + \Phi_{21} x_{1,t} + \Phi_{22} X_{2,t}\right)\Big] + \frac{1}{2}\Psi'_{2,\tau-1}\left(\Sigma_0 + \Sigma_1 x_{1,t}\right)$$

$$\Psi_{2,\tau-1} + \frac{1}{2}\left(\delta_{10} + \delta_{11} x_{1,t}\right)\left(\psi_{1,\tau-1} + \Psi'_{2,\tau-1} C_{21}\right)^2 \Big\} \qquad (38)$$

The gross expected rate of return on a τ – period bond after one period is this expectation $E_t\left[P_{\tau-1,t+1}\right]$ divided by its current price $P_{\tau,t}$. Taking the natural logarithm expresses this as a percentage return and subtracting the implicit one-period yield $y_{1,t}$ then gives the expected excess return or risk premium:

$$\rho_{\tau,t} = \log E_t\left[P_{\tau-1,t+1}\right] - \log\left[P_{\tau,t}\right] - y_{1,t}$$

$$= x_{1,t}\left[\Psi'_{2,\tau-1}\left(\Phi^Q_{21} - \Phi_{21}\right) + \psi_{1,\tau-1}\left(\xi^Q - \xi\right)\right]$$

$$+ \Psi'_{2,\tau-1}\left(\Phi^Q_{22} - \Phi_{22}\right) X_{2,t} \qquad (39)$$

$$+ \Psi'_{2,\tau-1}\left(\Theta^Q_{22} - \Theta_{22}\right) + \psi_{1,\tau-1}\left(\theta^Q - \theta\right) \qquad (40)$$

(using (36) and (36)). The risk premia implied by model M3 for representative maturities are shown in figure 8.

Appendix 3 : The Likelihood Function

This appendix derives the likelihood function and describes the numerical optimization procedure. Because the macro and measurement errors are assumed to be orthogonal, it shows that the likelihood of the joint model is the sum of macro and measurement components. Using (4) (9) and (18):

$$\begin{bmatrix} w_t \\ u_t \end{bmatrix} = A \begin{bmatrix} \upsilon_t \\ e_t \end{bmatrix} \sim N\left(0_{n+k}; F_t\right); \quad \text{where:} \qquad (41)$$

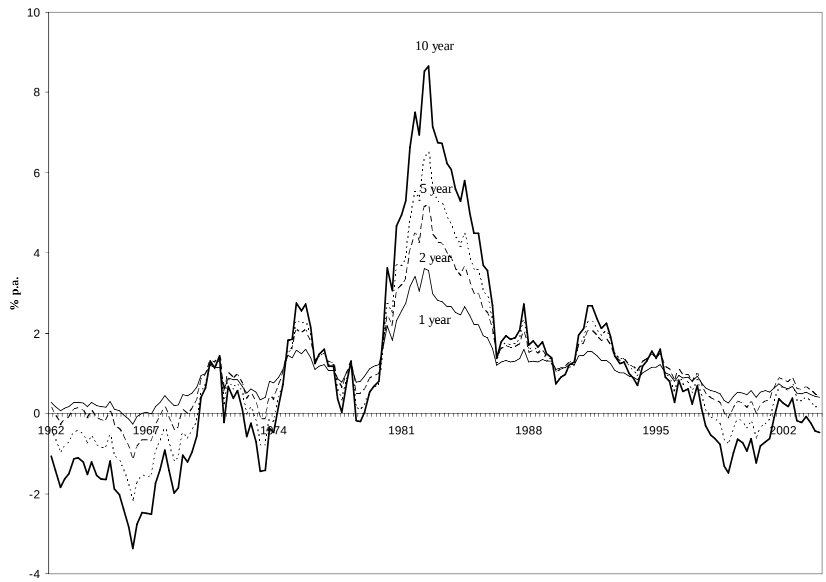

FIGURE 8.— Risk Premia in model M3

$$F_t = AD_t A'; \; A = \begin{bmatrix} \Gamma & 0_{n,k} \\ B\Gamma & I_k \end{bmatrix}; \; D_t = \begin{bmatrix} \Delta_t & 0_{n,k} \\ 0_{k,n} & \bar{P} \end{bmatrix}.$$

so the loglikelihood for period t can be written as:

$$L_t = -\frac{n+k}{2}\ln(2\pi) - \frac{1}{2}\ln(|F_t|) - \frac{1}{2}[\upsilon_t' \; \upsilon_t'] F_t^{-1} \begin{bmatrix} \upsilon_t \\ u_t \end{bmatrix} \qquad (42)$$

The Gaussian stochastic framework means that this likelihood function normalizes the one-period ahead squared prediction errors using the oneperiod ahead conditional variances in the usual way. However as in E(22) of Duffee (2002) this function can be expressed in terms of the macro and measurement errors using (41):

$$L_t = -\frac{n+k}{2}\ln(2\pi) - \frac{1}{2}\ln(|D_t|) - \frac{1}{2}[\upsilon_t' \; e_t'] D_t^{-1} \begin{bmatrix} \upsilon_t \\ e_t \end{bmatrix} \qquad (43)$$

$$= -\frac{n+k}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\ln\left(\delta_{i0} + x_{1,t-1}\delta_{i1}\right) - \frac{1}{2}\nu_t'\left(\Delta_0 + x_{1,t-1}\Delta_1\right)^{-1}\nu_t$$

$$-\frac{1}{2}\sum_{\tau=1}^{k}\ln(\rho_\tau) - \frac{1}{2}e_t'\overline{P}^{-1}e_t.$$

Summing this over T periods gives the loglikelihood for the estimation period:

$$L = -\frac{T(n+k)}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{n}\ln\left(\delta_{i0} + x_{1,t-1}\delta_{i1}\right) - \frac{T}{2}\sum_{\tau=1}^{k}\ln(\rho_\tau)$$

$$-\frac{1}{2}\sum_{t=1}^{\mathrm{T}}\nu_t'\left(\Delta_0 + x_{1,t-1}\Delta_1\right)^{-1}\nu_t - \frac{1}{2}\sum_{t=1}^{T}e_t'\overline{P}^{-1}e_t.$$

Since this is a quadratic in the (inverse) variances of the measurement errors (ρ_1, ..., ρ_6) in (17) this can be concentrated in the usual way by solving for their optimal values and substituting back. (This cannot be concentrated with respect to the parameters of the macro variances since these also affect the factor loadings in the yield equations.) This likelihood function was maximized using the FindMinimum numerical optimization package on Mathematica. This program checks that the admissibility restrictions (10) hold. It also eliminates observations for which the parameter estimates and data return negative variances, following Chen and Scott (1993). The estimates reported here include all observations, with non-negative definite variance structures.

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