

Some Important Issues Involving Real Options: An Overview

Gordon Sick*

University of Calgary, Canada

Andrea Gamba[†]

University of Verona, Italy

This paper provides an introduction to real options, as well as highlighting some important issues that are often neglected by real options analysts. While many books and surveys have been written on real options, there are some ubiquitous concepts that are not well-understood by many authors and practitioners. The objective of this paper is to redress this shortfall.

The paper discusses organizational issues that impede adoption of real options strategies. It discusses modelling and analytic techniques for real options. (JEL: C61, D92, G31)

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I. Introduction

There are several introductions to real options, such as technical books by Dixit and Pindyck (1994), Trigeorgis (1996), and practitioner books by Amram and Kulatilaka (1999) and Copeland and Antikarov (2001). There are several good surveys of real options, including Dixit (1989a), Pindyck (1991), Sick (1989), and Sick (1995). We assume that the reader may be familiar with some of these publications and they certainly contain the technical foundations of much of what we discuss here, so we will not provide detailed references throughout this paper. This paper is partially self-contained inasmuch as the key ideas are

* Haskayne School of Business, University of Calgary. This research was supported by a grant from TransAlta Energy Marketing. [†] University of Verona, Italy

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largely developed within the paper but the interested reader should know that these other papers and books are good sources for further exploration of these ideas. Our purpose in this paper is to provide a brief introduction that highlights several important points that we think have been neglected in the literature and by practitioners.

In the next two sections, we deal with conceptual issues and only discuss formulas where necessary to make a point. For readers unfamiliar with the formulas, references are made to later sections and references for the formulas, but readers can capture the important points of those sections without the detailed formulas.

Section II is a general discussion of four common characteristics (real assets, risk, leverage and flexibility) of real options and how they come together to create value for their owner.

Section III discusses organizational impediments to the implementation of real options strategy. They are generally agency problems. Some problems arise from the lack of urgency to immediately solve a real options problem so that they can become neglected. Other problems arise because real options strategy is often not a measurable activity as required by many incentive compensation systems. Such incentive compensation systems are at odds with shareholder value maximization because real options strategy is needed by any firm that wants to maximize value. Another problem arises from pro-cyclical investment strategies arising from capital rationing and balanced budget principles. Earnings smoothing is very popular in managerial circles and this is also hostile to optimal real option strategy.

Section IV shows how real options can be modelled as decision trees and how decision trees break down into two components: an influence diagram to model risk and a lattice to model risk. Discussion of the lattice approach is covered in greater detail in section VII.

Section V introduces popular mathematical approaches to the analysis of real options strategy and value: analytic solutions, numerical solutions of partial differential equations, lattices and simulation. The most practical of these for real options practitioners are lattice and simulation methods and both are discussed in greater detail in sections VII and VIII. Section V also introduces the notion of convenience dividends and discusses various types of stochastic processes, such as additive and lognormal Brownian motion, and Ornstein-Uhlenbeck and hybrid mean reversion.

Analytic solutions for real option analysis are based on the fundamental partial differential equation (PDE) for real option value.

These are discussed in section VI. It also discusses the smooth pasting condition for optimality of decisions.

Section VII discusses the lattice or tree approach to valuation of real options. It describes the settings for the magnitude of lattice jumps to model volatility and the assignment of risk-neutral probabilities to model drift.

Section VIII discusses the least squares Monte Carlo (LSM) method of assessing real options. It shows how to take a forward-looking technique like simulation to estimate conditional expected values, just as the lattice approach does. These conditional estimates are then compared for the various strategy alternatives and the result is carried in a backwards recursion to earlier times.

These latter two sections use the Bellman principle of optimality rather than smooth pasting to find optimal strategies.

Section IX offers some concluding remarks.

II. How Do Real Options Create Value?

Properly managed options create value and reduce risk for the organizations that own them. They arise because of the interplay of 4 things:

- (1) Real assets: financial options are generally redundant and hence do not create or destroy shareholder value. Real options cannot be replicated by stakeholders and generally create value.
- (2) Risk: volatility and risk-return relationships.
- (3) Leverage: variable costs and benefits work against either fixed costs and benefits or imperfectly correlated costs and benefits.
- (4) Flexibility: to manage the risk and leverage by accept upside risk potential and reducing downside risk.

A. Real Assets

Real assets include many things, such as real estate, factories, mines, oil wells, research & development and other intellectual property. Real assets are created naturally. In fact, real options analysis is often done

without a physical asset. We only need well-defined states of nature. For example, a tourist going to a seaside resort may be unsure of the destination weather and will bring a swim suit, rain gear and sailing gear to accommodate many possible outcomes of weather. The tourist is behaving in a sensible way to maximize the value of a real option on the vacation weather. Each day, the tourist will consider the weather forecast and choose an activity that suits the weather and the location.

In contrast, financial assets are an invention of humankind. They include stocks, bonds, and financial derivatives, such as options, futures, and swaps. These are generally contractual claims to real assets or outcomes that specify circumstances under which one party conveys the benefits of a real asset to a buyer of the financial asset. The literature on financial derivatives is enormous and many companies expend a lot of resources (personnel, trading fees, maintenance of trading desks) to manage their derivative portfolios.

Real Options Can Create Value Where Financial Options Cannot

The standard literature on financial derivatives is based on the observation that a financial derivative is redundant inasmuch as its payoff can be replicated by a portfolio of the underlying asset and some other asset, such as a bond. This analysis was popularized by Black and Scholes (1973) and is still the dominant methodology for assessing financial derivatives.

But, this begs a fundamental question: If the financial derivative is redundant, what value is created by trading the derivative? In fact, this is a serious issue that dogs the whole business of financial derivatives.

Here is another way to look at the issue. Some of the most fundamental results in corporate finance are the Modigliani-Miller (M-M) propositions. These propositions establish the irrelevance of corporate debt policy decisions and the irrelevance of the firm's dividend policy decisions. A simple extension of these results also establishes the irrelevance of the firm's hedging policy with financial derivatives. To see why, note that the M-M propositions essentially say that corporate financial decisions are irrelevant because they are undertaken by transactions in financial markets. Since financial markets are efficient, these transactions do not create or destroy shareholder value. All they do is rearrange the financial risk of the firm or the dividend payout of the firm. Now, shareholders can replicate the same financial policy, or undo the financial policy by making the same

transactions in financial markets themselves. Since the shareholders transact in efficient markets to adjust the financial exposure they have from the corporation, these financing transactions neither create nor destroy value.

Thus, we have a very fundamental version of the M-M propositions: When a firm deals with financial derivatives, it does not create or destroy shareholder wealth, because shareholders can costlessly do or undo these decisions on their own account.

Now, consider corporate decisions involving real assets. These include capital budgeting decisions to acquire, abandon, expand and contract operations. They include the decisions of when to turn plants and production systems on and off and what feedstocks to use or output streams to produce. These real asset decisions cannot be undone by a shareholder. A shareholder who thinks the firm has erred by developing a project too early cannot undo the transaction in financial markets. The firm has undertaken the decision and the shareholder can't reverse it. Similarly, if the firm should abandon a losing division, the shareholder can't replicate that decision by selling the division. Selling shares in the firm doesn't replicate the decision because the damage has already been done by the bad decision and the shares would be sold at a depressed value.

Thus, on a very fundamental basis, decisions involving real assets are relevant to creating or destroying shareholder value. To put it another way:

The firm can create significant shareholder wealth by having good real option policy and it can destroy significant shareholder value with bad real option policy.

Real Options Often Cannot be Valued by Replication Analysis

A real option to develop a project cannot be valued by a replication analysis if there is no market already in existence for the underlying asset. Thus, if the result of a real option strategy is to create a specific mine that produces a particular blend of minerals, there may be no pure play in a mine that provides the same risky payoffs as the yet-to-be developed mine. One cannot trade shares that haven't been issued. Real options to perform research and development typically depend on an underlying asset that does not exist until the real option is exercised.

Moreover, the risks that drive a lot of real options are quantity risks rather than price risks, such as the uncertain demand for petroleum,

electricity or transportation services or the uncertain supply of rainfall. Typically, there are few securities around that allow one to trade quantity risks.

Another problem is that real options have more types of stochastic processes than are often encountered with financial options. Financial options on stocks are usually modelled with a lognormal diffusion and a constant dividend yield. This is because the discipline of the market prevents some unusual behaviour, such as mean reversion. However, with real options, the dividend is often an implied convenience dividend, so market discipline does not rule out things like mean reversion.

Many commodity markets exhibit mean reversion because high prices are met by gradual supply and demand adjustments as suppliers build the capital equipment needed to increase supply of the good and consumers make capital adjustments to economize on the use of the expensive good. Thus, high prices tend to fall gradually towards a long-run mean. Similarly, low prices tend to gradually increase towards a long-run mean.

In contrast, suppose a stock price is mean reverting. Then, if the stock price is above its longrun mean, a profitable speculation will be to sell the stock short and cover the short position with a repurchase as soon as the price falls. Since short sales are relatively easy to make, there will be significant selling pressure on the stock until it falls to its long-run mean. The stock price will immediately, rather than gradually, fall to the long-run mean. A stock price below its long-run mean will give buying pressure until the stock meets its long-run mean. Thus, we do not see the gradual mean-reversion adjustments in stocks that we see in some commodities.

B. Risk

It is very uncommon to have a real option without some underlying risk driver.¹ The important characteristic of risk, or variation in economic

1. Indeed, some would say that risk is an essential feature of real options. That may be true, but some operating strategies can be driven by predictable seasonal variations in a fundamental asset price or business condition, and they have all the characteristics that we would ascribe to a real option. For example, a gas storage facility works on the seasonality of natural gas prices. The decisions of when to inject gas and when to withdraw gas are real option decisions and they are important even if there is no risk that gas prices will change from a seasonal pattern.

conditions, is that it requires dynamic decision making. The manager acquires information over time and uses it to make decisions about how to operate a business.

In the presence of risk, the manager will not be able to formulate a static “business plan” that describes what the organization will be doing at each date in the future. When companies, organizations or governments make headlines that say they will expand a plant starting 3 years from now, or build a traffic intersection 5 years from now, they are likely not following a real options strategy. They are likely spending out of free cash flow in a capital rationing context. If instead, they say things like “If ethylene prices reach \$ x then we will build another ethylene production plant,” or “If the intersection gets y cars per day, then we will upgrade the intersection,” then the manager may be following a real options strategy.

This means that an important feature of real options analysis is the modelling of risk. This generally requires the modelling of stochastic processes, and the following issues typically become important:

Drift or growth in the stochastic variable is important. This is the first moment in the variable representing increments or changes to the process.

Volatility of the process must be modelled. This is the second moment of the process.

Correlation between the stochastic processes underlying the real option is important because it determines the extent to which two processes combine to magnify risk or reduce risk in a natural hedge. For example, an oil company produces natural gas and crude oil and the prices of these two outputs are positively correlated, which increases risk. But, an electric power generator who burns natural gas has uncertain electricity prices as an output and uncertain gas prices as an input. These are positively correlated and the result is a natural hedge that reduces risk.

Correlation between the stochastic processes driving the real option and the stochastic processes that drive the systematic risk in the economy. This results in a systematic risk premium that must be modelled if the underlying risk cannot be replicated by a dynamic

Similarly, an airline has seasonal variations in its demand and a real options strategy response is to have new fuel-efficient planes running on long-distance “base-load” routes, where the plane is in the air a large fraction of the time. The airline can employ old planes (with a low capital value) on short routes or routes that may be cancelled in low season. This strategy makes sense even if there is no risk that demand will vary from the seasonal forecast.

trading strategy in existing financial products. The presence of this risk premium distinguishes real options from operations research strategies, which typically do not model such a risk premium, and from financial option strategies, which sidestep the risk premium by using a replication strategy.

Complexity of the risk model means that some of the standard approaches to analyzing financial derivatives will not work. Complexity arises from mean reversion, multiple risk processes, options to exchange one risky asset for another and jumps or discontinuities. For simple processes, the numerical solution of differential equations or the use of lattices or trees to represent the risk is often adequate. For more complex processes, the solution may be to use simulation, which is difficult in an optimizing environment.

C. Leverage

Many people will be surprised to see leverage on our list of key defining characteristics of real options, but we cannot imagine many real options that do not have leverage. The leverage in a development real option comes from the capital investment cost. The leverage in an operating real option comes from fixed operating cost.²

Sometimes the leverage comes from another stochastic value, when we have the option to exchange one asset or cash flow for another. For example, we can model a real estate development option in terms of a payoff that gives (re)developed urban land of uncertain value in exchange for surrendering underdeveloped urban land or agricultural land, either of which have uncertain value. We can have operating options where we take an input of uncertain value (e.g. natural gas or chemical feedstock) and exchange it for an output of uncertain value (electricity or plastic). As long as the two processes are not perfectly correlated, we have operating leverage.

2. There is also financial leverage arising from debt obligations. While this creates similar issues to those we discuss in this section, it is common to analyze real options on an unlevered basis, as if they were only equity financed and have only operating leverage. However, Aranda León et al. (2008) point out that there can be interest tax shields and bankruptcy effects associated with financial leverage and explore their impact on real option valuation. We do not explore this issue here, except to note that they find that real options even in an unlevered setting should be discounted at a lower rate from debt to account for preferential personal tax treatment of equity. They also explore how these tax shields are also offset by bankruptcy costs. Some of these issues are discussed in discrete time by Sick (1990).

Abandonment options involve the exchange a project of uncertain value for a certain abandonment value.³ The uncertain project value working against the fixed abandonment value also gives operating leverage. A useful summary variable that describes leverage is the ratio P/W of the value of the underlying asset value P to the value of (real) option value $W(P)$. Leverage combines with the hedge ratio $\Delta(P) = \frac{\partial W}{\partial P}$ to give the elasticity of the option price with respect to the stock price:

$$\eta(P) \equiv \frac{P}{W} \frac{\partial W}{\partial P}.$$

This means that a 1% change in the underlying asset value P leads to an $\eta\%$ change in the value of the option. Now, the traditional beta, β_p , is the measure of systematic risk in the underlying asset. It is traditionally measured by regressing rates of change in the underlying asset price on the systematic (market) index, so it is a measure of risk per dollar invested. Similarly, the volatility of the underlying asset, σ_p is a measure of standard deviation of rates of change in the underlying asset price over a unit of time, so it is a measure of total risk per dollar invested. Since these are measures of risk per dollar invested, we can multiply them by the elasticity $\eta(P)$ of the option price to get corresponding measures of risk for the option:

$$\sigma_w = \eta(P) \sigma_p = \frac{P}{W} \frac{\partial W}{\partial P} \sigma_p \quad (1)$$

3. Sometimes it is hard to identify the cash flow that is received when a project is abandoned. In practice, abandonment relieves the organization of the obligation to pay a stream of fixed operating costs that must be paid to get the output stream. The present value of this stream of fixed costs can be taken as the payout for an abandonment option, which is analogous to a put option in the financial world. In this case the asset that is surrendered is the claim to the stream of revenue.

In particular, note that the popular lognormal models of stock prices do not have a natural fixed value that is received upon abandonment. Without such a payoff, there is no incentive to abandon and we would never see firms in bankruptcy or projects being abandoned. This highlights the folly of modelling stock prices as being lognormally distributed. They are more likely lognormal less some cost that generates operating leverage. An abandonment analysis would take a lognormally distributed present value of the revenue stream as the underlying asset and the present value of fixed operating costs as the exercise price that is received upon abandonment.

$$\beta_w = \eta(P) \beta_p = \frac{P}{W} \frac{\partial W}{\partial P} \beta_p. \quad (2)$$

Equation (2) is particularly important as it lets us determine whether we can use a cost of capital approach to assessing real option values and strategy.⁴ If the elasticity $\eta(P)$ is constant as P changes, then we have a constant cost of capital for the option that we can compute from the capital asset pricing model (CAPM) or arbitrage pricing theory (APT). The risk premium over the riskless rate of return is proportional to beta in these models. On the other hand, if the elasticity $\eta(P)$ changes with P , as it usually does, then we do not have a useable cost of capital for the option. This is the reason why we generally cannot use risk-adjusted discount rates (RADRs) to analyze real options, and are, instead, forced to use a certainty-equivalent analysis involving risk-neutral probabilities.⁵

Now, the important thing we want to point out is that the main source of variability in the option elasticity is the option leverage in many cases. To examine this further, note that for a typical finite-lived development (call) option, as shown in figure 1, the hedge ratio $\Delta(P)$ is increasing from 0 to 1 as P increases from 0 to ∞ , since it is the slope of

4. This equation actually appears in Black and Scholes (1973), where they gave an alternative derivation of their option pricing model based on the CAPM. This derivation is a more useful approach in real option pricing models than the replication approach.

5. There is an interesting knife-edge situation where the hedge ratio effect exactly matches the leverage effect to leave a constant cost of capital for a real option. It is the situation of a perpetual american option on a lognormally distributed underlying asset, discussed in section VI, A. Differentiating formula (14) for the value of the development option (where $A_2 = 0$), we have

$$W(P) = A_1 P^{v_1}$$

$$\Delta(P) = \frac{dW}{dP} = v_1 A_1 P^{v_1-1}$$

$$\frac{P}{W} = \frac{1}{A_1 P^{v_1-1}},$$

so that the elasticity

$$\eta(P) = \frac{P}{W} \Delta(P) = v_1.$$

which is a constant. Thus, these perpetual real options on a lognormally distributed underlying asset have a constant beta and can be evaluated by risk-adjusted discount rates.

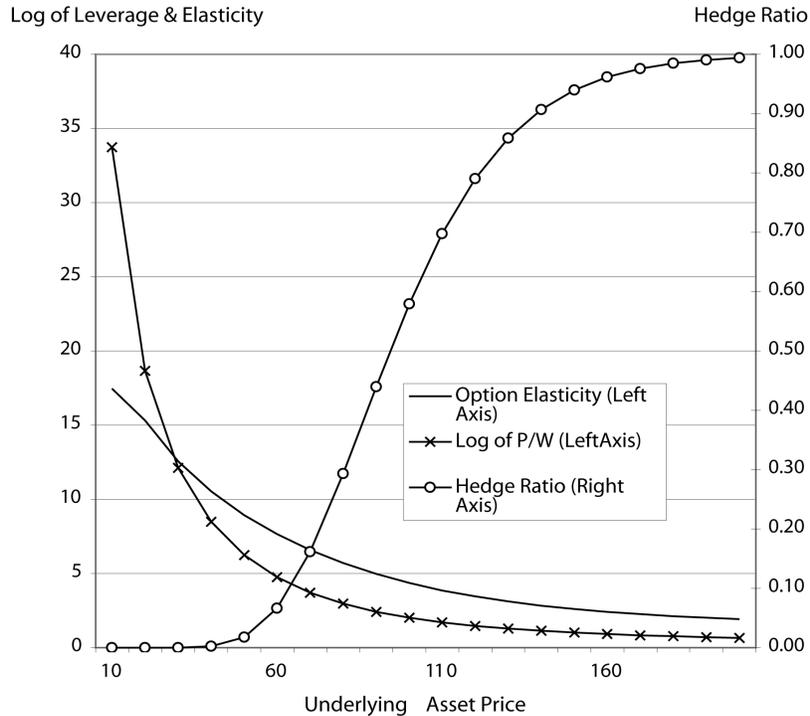


FIGURE 1.— The elasticity of the option is the product of the leverage ratio and the hedge ratio. The leverage ratio varies much more strongly with underlying asset price so that the elasticity and option beta increase as the underlying option price falls. This is for a european at-the-money call option with one year to maturity, a volatility of 30% and a discount rate of 3%.

the option value, which is a convex, increasing function. Note, also, that the leverage ratio P/W decreases as P increases. Indeed, we may recall from the kinds of leverage ratio arguments used in an introductory finance course, the leverage ratio of an option at maturity blows up to ∞ as P falls from above towards the development (exercise) value K . Thus, while there is the potential for the decreasing leverage ratio to exactly offset the increasing hedge ratio, leaving the elasticity and option beta constant as P increases, it doesn't happen in this common case.

The message here is that we cannot count on an option having a

constant cost of capital,⁶ so we are forced to use risk-neutral (certainty-equivalent) pricing. This problem is partly due to the fact that option hedge ratios change with the underlying asset price, but the major source of variation in the option beta is actually the operating leverage coming from the fixed development costs. This problem occurs in valuing risky assets even in the absence of flexibility and real options. Modern asset pricing is forced to consider more highly levered cash flow streams with swaps, spreads and real options. It is time to give much more serious consideration to risk-neutral or certainty equivalent pricing in these situations. This is commonly done in financial derivatives, but the principle should be employed in many more real asset situations.

D. Flexibility

The fourth important characteristic of real options is flexibility. Without flexibility, all we have is a complex risk model – the real option cannot create value. While this is trivially true, it is easy to overlook.

Flexibility means that the manager must make decisions contingent on the information that arrives about the risk variables. It means, in particular, that the manager cannot have fixed or static plans. Common business plans are static. Organizations often post 5-year plans of capital investment. When an organization slavishly adheres to such a fixed plan, it loses the ability to use flexibility, which is needed to create real option value.

How Do Real Options Create Value and Manage Risk?

Real options combine flexibility, leverage and risk to create value. One way to consider this is to note that the flexibility allows the manager to capture upside potential while mitigating much of the downside risk. The typical development option to receive an underlying operating project of uncertain value P by paying a capital construction cost K pays off with “hockey-stick payoff” $\max\{0, P - K\}$. The payoff is shown in figure 2. The payoff function is a convex function, and we typically get convex functions as the result of optimizing decisions. For example, if

6. See Hodder et al. (2001) for a further discussion of this problem with a stochastic cost of capital.

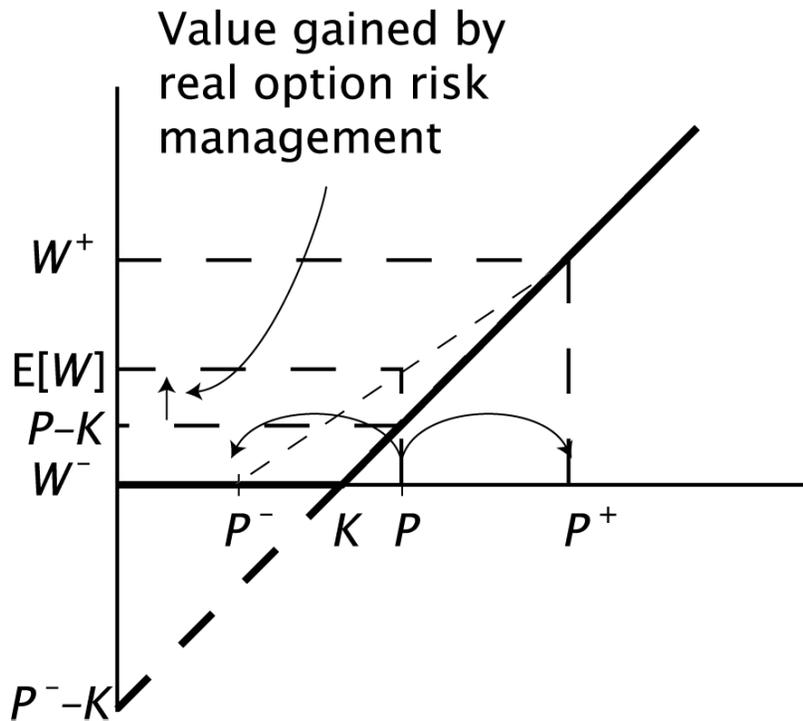


FIGURE 2.— Real options create value by mitigating downside risk.

we do not have the right to delay the project, the payoff is $P - K$, which is the dotted straight line that is not convex. An important mathematical result involving convex functions is Jensen's inequality. For a function $W(P)$ of a risky variable P , it says that the expected value of a convex function is greater than the value of the function evaluated at the expected value of the underlying variable:

$$E[W(P)] > W(E[P]).$$

Figure 2 shows a situation that starts with an initial underlying asset value P . The asset price can go to P^+ or P^- with equal probability before the final payoff. Since the payoff is truncated at 0, the option payoff in the down state $W^- > P^- - K$ so the option mitigates downside risk. The expected option value $E[W]$ is thus above the value of the option payoff $P - K$ that would occur without the risk.

Alternatively, we can think of real options as providing a

risk-management strategy. By using the flexibility of waiting to make decisions until more is known about the potential payoffs, the manager can reduce the downside risk. A european option will truncate the left tail of the probability distribution at the exercise price.

III. Impediments to the Adoption of Real Options Strategy

The following issues often arise to induce managers to avoid a value-creating real options strategy:

- (A) The smooth pasting condition. Not misunderstanding it, but the lack of discipline that it encourages.
- (B) Activity-based compensation systems that encourage management to exercise their real options too early.
- (C) Pro-cyclical investment policies by governments and corporations as they balance budgets and smooth revenues.
- (D) Earnings smoothing by managers.

We discuss each of these issues below.

A. *The Smooth Pasting Condition*

The smooth pasting condition is an optimality condition that describes the optimal trigger point of for development of an american (real) option. Figure 3 shows the solutions to an american call option with various trigger values P^\dagger , where the trigger value is the underlying asset value that triggers the development decision.⁷ At the optimal trigger point, the graph of the option value touches the payoff function and the two graphs are tangent to each other. If the option is exercised too early, such as at $P^\dagger = \$160$, the option value lies below the payoff boundary $\max\{0, P - K\}$ for $\$160 < P < \575 . Choosing a payoff trigger between

7. The graph is for a perpetual american call option on a lognormally distributed asset when the development cost is \$100, the discount rate is 3%, the dividend yield is 3%, and the volatility is 25%. The solution to this problem is given in section VI, A, and the optimal trigger value is given by equation 17 as $P^{C*} = \$266$.

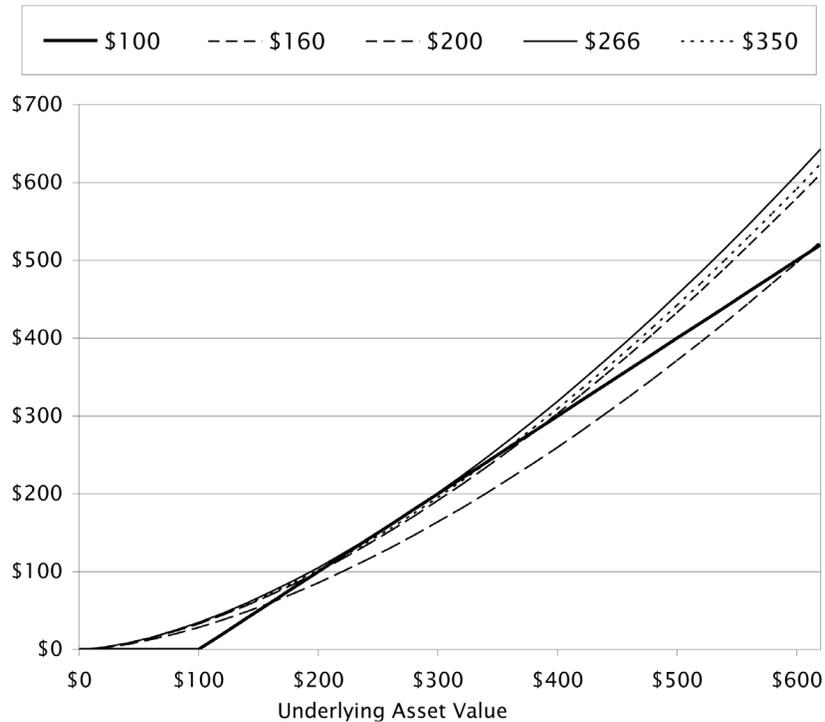


FIGURE 3.— This graph shows real option values as a function of the underlying asset value for various choices of the exercise trigger P^\dagger . The real option is optimally developed at a trigger of $P^* = \$266$. The value associated with other development triggers is also shown, along with the intrinsic value, $\max\{0, P - K\}$, which corresponds to a trigger value from the traditional NPV rule, where $P^\dagger = K = \$100$.

these two points will increase option value. Thus, at the optimum, the option value must be tangent to the exercise payoff function.

This is fine, since smooth pasting can be used to help solve for the optimal real option value. The problem arises because one can exercise a real option a little too early (e.g. $P^\dagger = \$200$), or a little too late (e.g. $P^\dagger = \$350$) and the value of the option is not significantly impaired, as shown in figure 3. The significant loss in value occurs when the option is developed too early at prices between \$100 and \$200.

Figure 4 shows similar information in a 3-dimensional graph. We can see that significant value is lost by exercising very early (near the development cost, which is what the traditional NPV rule would have

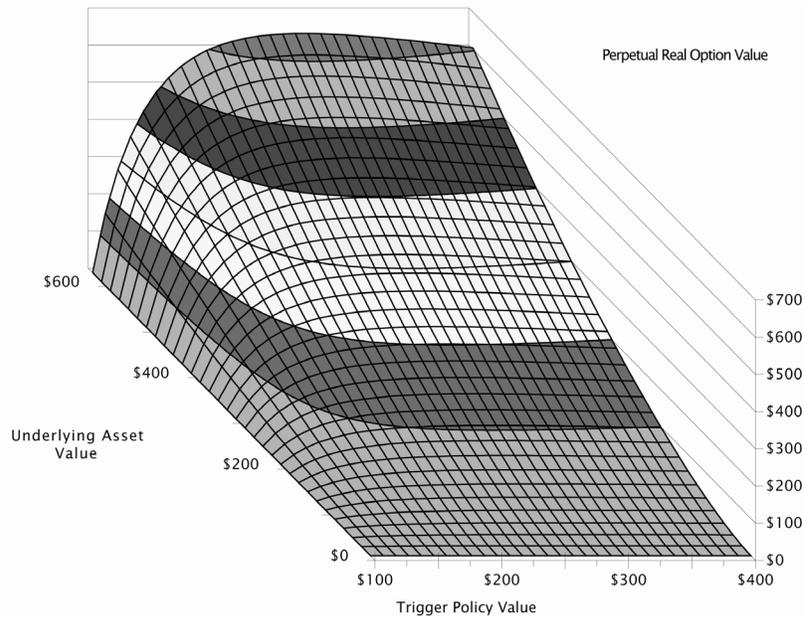


FIGURE 4.— The option value for various trigger hurdles.

suggested), but relatively little value is lost by choosing a broad range of trigger values that are near the optimal trigger value of \$266.

Why is this a problem? Well, there is no sense of urgency to get the exercise point precisely right. This makes it easy to defer dealing with the problem (exercise late). Exercising too late is just like procrastination — the job never gets done without a deadline. Thus, real option strategy can get confused with lethargy or procrastination.⁸

The smooth pasting condition gives us the luxury of robustness of the real option to not act in the precisely optimal way. But this same robustness allows us to be sloppy in analyzing real options. This allows us to use rules of thumb or other dangerous proxies for real option management that cause us to lose sight of the real problems.

For example, the oil industry is quite aware of real options, but few firms have any tools in place to optimize the value of real options. However, they regularly acquire tools to do “portfolio management” which is a complex mean variance analysis technique for selecting projects that is lacking in any theoretical or market-based

8. Later, we discuss the misplaced incentives to exercise too early.

underpinnings. Many companies use Value at Risk (VaR) as an analytic tool, which, despite its complexity, does nothing to help them create shareholder value. Thus, companies are not averse to using a complicated quantitative analysis, but they are unfortunately averse to using real options strategy.

B. Activity-based Compensation Systems

Incentive compensation must be based on something that can be measured objectively. Ideally, one would like to choose a compensation system that measures an individual's contribution to the value of the firm. This is the basis for value-based incentive compensation systems, such as Stern Stewart's Economic Value Added (EVA). In the right circumstances, these can provide the incentives for managers to maximize the value of the firm. There are other popular measures of performance that don't do so well. In order to be objective, these measures are often based on measurable characteristics such as activity. Thus, managers are often given compensation based on the level of sales (or sales growth), the level of profit, the number of employees they supervise or the dollar value of the assets that they administer.

Since a real option to defer often has little measurable activity associated with it until it is exercised, there is a strong tendency to not give incentive compensation to a manager who maximizes option value by deferring. Incentive-based compensation often induces managers to destroy real option value by exercising too early.

It is instructive to understand the circumstances under which compensating the manager with EVA induces her to maximize firm value. If the manager

- (1) lives forever,
- (2) doesn't leave her job, and
- (3) has the same adjustment or discount for risk as a well-diversified shareholder,

then she will manage and select projects to maximize the present value of the risk-neutral expectation of EVA-based incentive compensation. If her compensation is linear in EVA, this means she maximizes the risk-neutral expectation of the stream of EVA. If her discount for risk is the same as that of a diversified shareholder ("the market"), she has

the same risk-neutral expectation operator as does the market. If she lives forever in the same job, she looks at precisely the same stream of cash flows as does the market. Thus, in maximizing the value of the stream of her EVA, she maximizes the market value of the firm. Under these assumptions, EVA-based incentive compensation will induce the manager to maximize real option value. This is good, because it establishes that not all activity-based compensation systems necessarily destroy real option value.

However, if these three criteria are not met, we can have a wedge between manager and shareholder incentives. Perhaps we can develop EVA adjustments that mitigate these problems. Of course, we could develop whole new compensation systems. The problem becomes a principal-agent problem with an asymmetry between the information (agent) of the manager and the less-informed shareholders (principal).

C. Erroneously Using Pro-Cyclical Investment Strategies

Many companies regularly employ capital rationing techniques, which resemble real option techniques inasmuch as they force the firm to delay projects beyond the point where the NPV is simply positive. However, they get the details of the capital budgeting process dangerously wrong. Typically, the budget for capital rationing is set equal to the free cash flow of the firm, which is basically income from the prior year, plus depreciation less dividends and required capital expenditures. Thus, if the prior year's income is high, the capital budget is high and more projects are accepted.

This results in a pro-cyclical investment pattern that lags rather than leads the economy. A real options analysis requires forecasting to determine the value of an exercised project and leads the economy.

Similarly, we must be concerned with the popularity of budget balancing by governments all over the world. While budgets should be balanced in the long run, governments now balance them in the short run as well. This leads to the same pro-cyclical boom and bust spending. Pro-cyclical spending patterns just lead to a boom and bust economy, which does nobody any good.

Certainly governments of the 1960s and 1970s lacked fiscal discipline as they pursued a neo-Keynesian policy of injecting deficits into their economies to get them going faster, when the net result was really to get more inflation. The reaction in the 1990s for governments of all stripes around the world was to focus on balancing budgets. They balanced budgets on an annual basis, not even averaging over a

multi-year period. This is just a variation of corporations using capital rationing and spending out of free cash flow. It seems that politicians think that their electorates have so little faith in their government's ability to follow a disciplined capital investment strategy that they precommit to an irrational investment strategy — giving up their rights to manage investment properly. This results in a pro-cyclical investment strategy on the demand side (demand for investment goods): when the economy is booming and cash flows and taxes are high, they spend more, creating more of a boom. When times are bad, they cause more contraction of the economy.

A real options strategy of development and abandonment is characterized by a hysteresis effect: firms are less likely to start a project and less likely to abandon a project than a simple NPV analysis would suggest. Thus, there is a smoothing effect, rather than a boom-bust effect, in the deployment of projects. This assists in the planning of scarce resources, such as key personnel who are needed to manage or engineer projects. It results in smoother investment cycles.⁹

D. Earnings Smoothing Can Destroy Real Option Value

Managers and shareholders are taking very narrow views of firm performance these days. They focus on sequential earnings growth so that they can compare quarter-to-quarter earnings changes or year-to-year earnings changes. Perhaps the recent accounting scandals will defuse some of this excessively narrow focus by making investors understand the extent to which earnings can be manipulated with various accounting treatments.

However, many managers are still deluded to believe that smooth earnings really is a key component of share value. One of us recently heard a senior manager at an energy company, which had an electric utility division and a petroleum production division, who said that he couldn't really afford to adopt a real options strategy because it would cause him to defer projects sometimes and develop them at other times. The result is that his company's earnings stream would be more volatile and the market would regard his corporate revenue to be more heavily weighted towards the company's oil and gas assets than its electric utility assets. In other words, he thought that the oil and gas division would command a lower P/E multiple than would the utility division,

9. However, this investment smoothing doesn't necessarily smooth earnings or cash flow.

because of its higher risk. He thought he had to smooth earnings to make the oil and gas subsidiary resemble a utility, and hence command a higher multiple. If this can really be true, it is a sorry indictment of the poor quality of accounting information coming out about his firm.

Perhaps one good thing that will come of the accounting scandals is that investors will become inherently suspicious of activity as reported in accounting numbers and look to deeper understandings of whether the firm is being properly managed. Hopefully, investors will expect a meaningful real options strategy to be spelled out in the Management Discussion and Analysis (MDA) section of their Annual Reports.

Much of the OPEC member behaviour in setting and breaking oil production quotas is also related to revenue smoothing. That is, if oil prices fall, they sell more oil. If many OPEC members pursue this strategy, falling oil prices will cause overproduction. This drives oil prices even lower, creating a pro-cyclical supply response: oversupply begets more supply. The result is a boom and bust oil economy. While this problem occurs at the production level, it can easily extend to the level of capital expenditures. It is in direct odds to optimal real option strategy.

IV. Modelling Real Options with Decision Trees

One way to model real options is with a decision tree, which incorporates decisions by nature (that is the up-down moves of the underlying asset price) and decisions by the manager (developing, abandoning or adjusting production rates, for example) in a sequential manner.

- (1) Decision trees are based on a sequence of alternating decisions by a manager (denoted by a square node) and Nature (denoted by a round node as the probability of going up or down).
- (2) Decision trees are solved by working backwards from the tips of the tree at the farthest date.
- (3) Managerial decisions are made by taking the the most highly valued decision at the node, and the associated value is taken back to the preceding node.

Optimal decisions are bold italic and paths are bold.

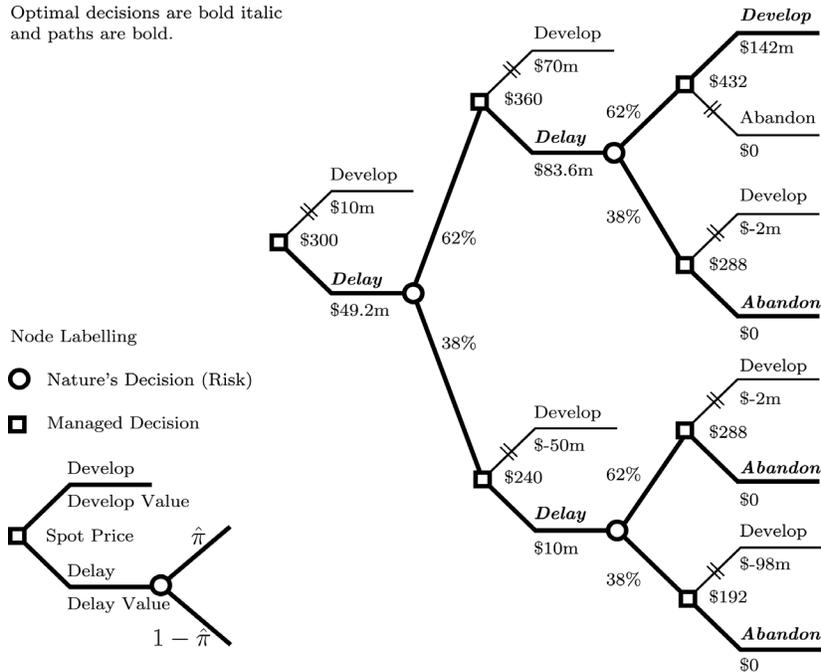


FIGURE 5.— Decision tree to develop a gold mine.

(4) Nodes corresponding to risky choices by Nature are valued by taking the expected value from the two alternatives and discounting for the time value of money.

A real option decision tree to develop a gold mine is shown in figure 5. It is based on an initial gold price of \$300 per ounce and reserves of 1 million ounces of gold. The gold can all be immediately produced for a combined capital and operating cost of \$290 million. Gold prices will rise by 20% over a year with a probability of 62% and they will fall by 20% with a probability of 48%.¹⁰ The riskless discount rate is 6% and the company can develop now, or defer for either one or two years, after which point the opportunity to invest is lost.

If immediately developed, the NPV is \$10 million. But the optimal value of the real option is larger: it is \$49.2 million. This is determined in the decision tree in figure 5. The only situation in which the mine

10. We will go into greater detail on how to set these probabilities in section VII.

would be developed is at date 2 if the price rises twice. In all other situations, it is optimally delayed or (at date 2) abandoned. The extra value accruing to the real option over the NPV is that the delay allows the manager to learn information and reduce downside risk, while retaining upside potential.

This decision tree models a very simple problem with two periods and only a decision of “develop” or “delay” for the manager. Yet the decision tree is quite complex and time consuming to build and analyze. We clearly would like to put more time periods into realistic problems and include more flexibility, including delay, abandon, expand, rebalance input vectors and output vectors.

The problem with the decision tree is that it combines information about Nature’s risk outcomes and the manager’s decisions. The manager would generally like to leave the risk modelling to a numerical analyst or a specialist in the construction of computational engines such as binomial lattices, numerical partial differential equations or simulation. We discuss these approaches in section V. The manager would prefer to leave that part of the analysis to someone else and focus on what is unique about the project at hand: the choices the manager can make. We discuss the manager’s approach next.

A. Modelling Flexibility with Influence Diagrams

The influence diagram focuses on the choices or flexibility that the manager has and leaves the risk analysis hidden. The manager only needs to describe the parameters of the risk process once and then leave that to a computational engine.¹¹

Figure 6 shows a influence diagram for our mine development option that is expanded to include abandonment. We have three operating modes:

- (1) Undeveloped
- (2) Operating
- (3) Abandoned.

11. This is not the only way to decompose a real option problem into smaller parts. Gamba (2003) uses a compound option approach, where the building blocks are simple call and put options. When a compound option is exercised, the owner gets another compound or simple option.

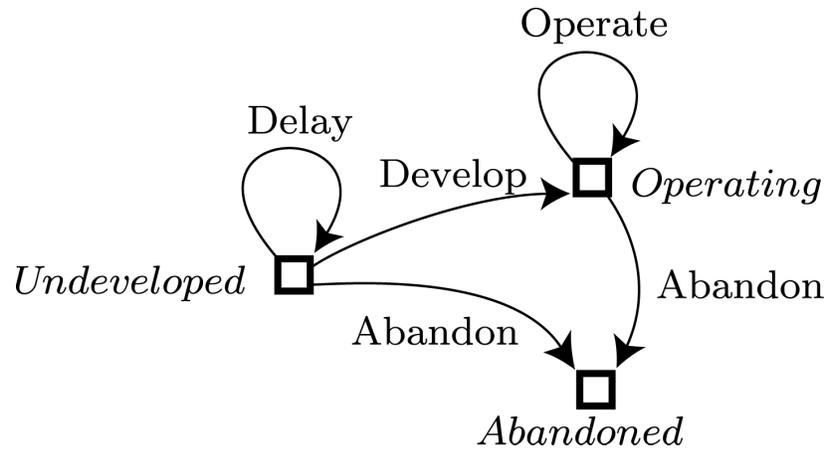


FIGURE 6.— The influence diagram for the gold mine development real option.

The curved arrow edges describe the transitions we can make:

- (1) Delay development
- (2) Develop
- (3) Operate a developed mine
- (4) Abandon a developed mine
- (5) Abandon an undeveloped mine.

Some information needs to be associated with each operating mode. The following information is relevant:

- (1) Cash payoff that will occur if the project ends while in this mode.
- (2) Whether this is a beginning or an ending mode.

Most of the modelling in the influence diagram occurs with the transitions. Associated with a transition is the following information:

- (1) Criteria under which the transition can occur.

(1.1) Perhaps the transition can only be made early or late in the project life, or perhaps it cannot be made unless some random variable is in some target range.

(1.2) Some of these constraints are not binding, given the desire to optimize the value of project, so they can be ignored.

(1.3) For example, it may seem to make sense to prevent a decision to enter a phase to sell a product if the random demand for the product is not positive, but this is not likely to be a binding constraint if it is possible to do nothing, which is a more valuable decision.

(2) Cash flows associated with the transition.

(2.1) These can be lump-sum costs, such as an development costs, or flow costs that occur throughout the transition time period.

(2.2) The cash flows can be a function of stochastic variables, such as spot prices that represent revenues or expenses or stochastic quantity variables representing uncertain demand or supply.

We model the manager's flexibility by the constraints and cash flows of the constraint edges, as well as the source and terminal modes of each transition edge. Not all modes have to be connected directly to each other and irreversibility is represented by a one-directional transition edge.

V. Approaches to Assessing Real Options Strategy and Value

There are four basic computational methodologies for assessing real options value and strategy. They are compatible with each other, but in different situations, one technique may be more useful than another. The focus has to be on using the simplest technique that gives useful results. Few organizations have the tolerance that academicians have for precisely correct solutions to unimportant problems. On the other hand, organizations should be denied the excuse that real options are too complex to be worth implementing. It is clear that real options strategy is essential to unlock the full value from modern organizations.

Many of the simplest formulas are for european options, which are options that can only be exercised at a specific maturity date. Unfortunately, most real options are american options, which require the manager to decide when to exercise the real option. Solving for the early exercise criterion is often the biggest part of the problem. For simplicity, we will use the traditional terminology “call” for a development option and “put” for an abandonment option. There are many other types of real options, such as options to manage a plant or resource extraction rate.

Here are the four basic methodologies:

Closed-form analytic solutions are the best approach to use when they are available. These include the Black-Scholes formulas for european put and call options, and the solutions for perpetual american put and call options on normally or lognormally distributed underlying assets. Most real options do not fit these categories perfectly, but they are useful limiting cases and valuation bounds for some real options that do occur naturally. We will review these briefly.

Numerical solutions to partial differential equations (PDEs) are generally useful only in academic real option settings because the practitioners generally face too great a variety of problems to be justified by building a custom PDE solution for every real option they come across. There are some software tools coming out now to solve broad classes of real option problems and some of them employ solutions to PDEs, but the user never knows the solution algorithm, and doesn't need to know the details. We will leave this methodology aside in this paper because we don't have the time or space to study numerical solutions of PDEs, and a practitioner wouldn't need to know this to employ a software tool using PDEs.

Lattice or tree models for real options are useful because they are easy to understand and work well for american or european options. If there is only one risk driver, they can be implemented on a spreadsheet, where one axis is time and the other is the price level or value of the underlying risk driver. We will briefly review these models. Their limitations arise when we have multiple risk drivers. The solution to these models is generally too messy to implement in a spreadsheet and requires a programming language, or a numerical programming language like Matlab, Maple or Mathematica. The multi-dimensional lattice models approach the complexity of the numerical solution of PDEs. We will discuss lattices with one risk driver only. The optimal trigger strategy and option value is computed the backward recursion of

a decision tree, which is also known as the Bellman equation of dynamic programming. This replaces the smooth pasting condition of optimality.

Simulation models have been used for many years to analyze european options, but it was generally felt that they would not be useful in analyzing american options. This is because simulation is a forward approach, with the underlying asset starting at a fixed price and undergoing random increments going forward. On the other hand, the Bellman equation requires a backward recursion. However, we now know that simulation can be efficiently used to estimate the conditional expected payoffs and continuation values that would be used in a lattice approach. Moreover, simulation can easily handle multiple risk drivers and complex processes, which is a distinct advantage over the lattice approach. We will discuss the simulation approach as well.

A. Stochastic Processes for Real Options Risk

It is simplest to describe a stochastic process for the underlying “asset” price P_t by a diffusion equation:

$$dP = \alpha(P)dt + \sigma(P)d\omega, \quad (3)$$

where

$$\alpha(P) = \text{Drift or growth rate of } P$$

$$\sigma(P) = \text{Standard deviation for one unit of time } t$$

$$\omega_t = \text{Stochastic process.}$$

The stochastic process $\{\omega_t | t \geq 0\}$ could be a Poisson jump process or a variety of processes, but we will restrict ourselves to the assumption that it is normally distributed Brownian motion with zero drift and unit variance per unit time:

$$\omega_t - \omega_{t-1} \sim N(0,1).$$

If we have multiple stochastic processes, we model them with different underlying Brownian motions $\omega^1, \omega^2, \dots$ and we must specify the correlation between them $\rho_{i,j}$.

By, taking different functional forms for the drift and standard

deviation, we get common stochastic processes, such as:
Lognormal Diffusion

$$\alpha(P) = \alpha_p P \text{ and } \sigma(P) = \sigma_p P \quad (4)$$

Normal Diffusion

$$\alpha(P) = \alpha_p \text{ and } \sigma(P) = \sigma_p \quad (5)$$

Ornstein-Uhlenbeck Mean Reverting Process

$$\alpha(P) = \lambda(\mu - P) \ (\lambda > 0) \text{ and } \sigma(P) = \sigma_p \quad (6)$$

Hybrid Mean Reversion or Integrated Geometric Brownian Motion

$$\alpha(P) = \lambda(\mu - P) \ (\lambda > 0) \text{ and } \sigma(P) = \sigma_p P. \quad (7)$$

The lognormal diffusion and hybrid mean reversion process standard deviation is the product of a volatility σ_p and the price itself, which ensures that the price P never becomes negative if it starts with a positive value.¹²

The normal diffusion and Ornstein-Uhlenbeck (OU) processes both can give negative “prices” P , which is often a problem with models of financial derivatives, because assets are often assumed to have limited liability. However, the underlying risk driver for a real option does not need to be a marketed asset. Indeed, for a development option, there usually is no market for the asset until it is developed. This is another example where real options analysis is more general than financial options analysis.

Similarly, mean-reversion is generally a poor model for the price of a traded company share, because it would provide speculative arbitrage opportunities for an investor who sells the share short when its price is above the long-run mean ($P > \mu$), expecting to profit from the downward drift in price. Similarly, a profitable speculation is to buy when the share price is below the long-run mean ($P < \mu$). However,

12. It is not trivial to verify this for the hybrid mean reversion process, but Robel (2001) has shown this in a working paper. He calls it Integrated Geometric Brownian Motion (IGBM).

these are good processes for modelling many commodities, particularly metals and energy commodities because they are created and consumed by capital-intensive processes, and the capital adjustments when spot prices are above or below the long-run mean take time to build or depreciate. For example, suppose the long-run mean for oil price is \$35, but the spot price of oil is \$20. Then, petroleum companies will delay their drilling programs, knowing that it is better to wait until prices are higher. This creates a shortage of oil, which tends to gradually increase oil prices towards the long-run mean.¹³ Meanwhile, consumers of oil will buy cars that are less energy efficient, and build homes and factories that are less energy efficient. This increases demand and tends to gradually increase price as more and more of these energy-intensive consumption processes are put in place.

The profitable speculation that would take place for a mean-reverting stock does not take place for a mean-reverting commodity because there are storage costs and convenience dividends for the commodity that offset the anticipated speculative profit. When the commodity price is above its long-run mean, there is a significant convenience value or dividend to be earned by the holder of spot commodity that is not earned by someone who has a futures or forward contract that delivers the commodity at a later date. The convenience value represents the value of being able to use the commodity if there is a sudden need for it, or a sudden stoppage in the delivery system for it.

B. Convenience Dividends

Modelling convenience dividends is most easily discussed in an equilibrium context. Consider the traditional capital asset pricing model (CAPM) or single-factor arbitrage pricing theory (APT). For example, suppose we have a lognormal diffusion for the stock price, which pays a continuous dividend yield of δ and a beta of β_p . If the market risk premium is $\gamma = E[r_m - r]$ where r_m is the market return or APT factor (per unit time) and r is the riskless return, then the CAPM or APT says that the expected capital gains rate plus dividend yield equals the

13. The strength of mean reversion is the coefficient $\lambda > 0$. A useful interpretation for it is the half-life of mean reversion, which is

$$T_{1/2} = \frac{\ln 2}{\lambda}.$$

riskless rate plus a risk premium that depends on beta:

$$\alpha_p + \delta = r + \gamma\beta_p.$$

Rearranging this equation, we find two equivalent definitions of the risk-neutral drift rate $\tilde{\alpha}$:

$$\tilde{\alpha}_p = r - \delta = \alpha_p - \gamma\beta_p. \quad (8)$$

We call this the risk-neutral drift rate because it is the capital gains growth or drift rate that would be required by a risk-neutral investor to hold the stock. That is, the risk-neutral investor requires a capital gains rate of $r - \delta$, because, when added to the dividend yield δ , this provides the investor the riskless rate of return. We won't attempt to reproduce any of the common arguments about risk-neutral asset pricing, but we just point out that assets and derivatives or options on those assets can be correctly priced by using the risk-neutral distribution of returns to get riskneutral expected payoffs and then discounting those payoffs at the riskless rate of return. The risk-neutral expected payoffs are thus certainty-equivalents, so that modern derivative pricing is done using certainty-equivalents rather than risk-adjusted discount rates. Equation (8) shows that there are two equivalent ways of determining the risk-neutral drift rate for the lognormal diffusion:

- (1) If the dividend yield δ on the asset is observable, then the risk-neutral drift rate on a lognormally distributed asset is $r - \delta$.
- (2) If the dividend yield on the asset is not observable, but the systematic risk β_p and expected growth rate α_p are both observable, then the risk-neutral growth rate can be calculated as the true growth rate minus a risk adjustment: $\tilde{\alpha}_p = \alpha_p - \gamma\beta_p$.

Note how the approach implicitly defines the convenience dividend yield δ . This is important in real options where the underlying asset may be untraded and not have an explicit dividend. For example, we can use the second approach if P is the quantity sold of the product that might be produced when the real option is exercised. This is a quantity rather than a formal price, so the dividend is not directly observable. However, we may be able to model the growth rate in sales and the systematic risk

of sales, β_p , which is a function of the volatility and correlation with the systematic market factor.

Now, we will extend this to more general stochastic processes. Multiplying both sides by P gives a risk-neutral drift $\alpha_p P$ in terms of dollar flow:

$$\alpha_p P = rP - \delta P = \alpha_p P - \gamma \beta_p P.$$

Note how the systematic risk, in dollar terms is $\beta_p P$, and is proportional to the total risk, also in dollar terms: $\sigma_p P$. This is because the risk in P comes through the diffusion term $d\omega$ and that is the only source of joint variation with the systematic market factor. Thus, we can write the total systematic risk as $\beta(P) = \beta_p P$ and the general risk-neutral drift for the general diffusion (3) is thus:

$$\alpha(P) = \alpha(P) - \gamma \beta(P). \quad (9)$$

We will use this risk-neutral drift in the remainder of our discussion of real options analysis.

It is interesting to see what the implied convenience dividend is for our two mean-reverting processes:

$$\begin{aligned} \delta(P) &= rP - \alpha(P) \\ &= rP - \lambda(\mu - P) + \gamma \beta(P). \end{aligned}$$

For the Ornstein-Uhlenbeck mean-reverting process, beta takes the form of a constant (like the standard deviation) that is measured in dollars per unit time: $\beta(P) = \beta_0$. For the hybrid meanreversion process, the beta takes the traditional form of the lognormal diffusion: $\beta(P) = \beta_p P$.

VI. The Fundamental PDE for Valuing Assets and Real Options

Suppose we have a real option of value $W(P, t)$ that depends on an underlying asset of value P . It's value will depend on time t , as well, if it has a finite time to maturity (maturity decay effects). Suppose that the owner of the real option receives a dividend $D(P)$. For example, we may

have a real option to convert agricultural land to urban land, so the dividend paid to the real option owner is the net cash flow to be received on agricultural operations. In many cases it is zero. Then, we have the fundamental risk-neutral PDE for the valuation of the real option:

$$rW = D + \frac{\sigma^2(P)}{2} \frac{\partial^2 W}{\partial P^2} + \frac{\partial W}{\partial P} \alpha + \frac{\partial W}{\partial t}. \quad (10)$$

This equation has the following risk-neutral interpretation. The left side is the rate of return that a risk-neutral investor requires for an investment of value W . The right side describes the total return the investor expects to get, and the two must be equal in equilibrium. The first term on the right is the dividend payout that the owner of the option expects to get. The second term is the so-called Itô adjustment that reflects the drift in the value of W that the investor can expect from the interaction of the curvature of the function W with the variance of the underlying asset.¹⁴ The third term describes the growth in the option value that comes from risk-neutral growth in the underlying asset. The final term represents the direct growth (time-dependence) or decay in the option. The final term is zero for perpetual options, as there is no decay arising from the approach of the option to maturity.

To determine the value of a real option, we only need to describe the option payoff at some points, which are often called boundary points. Typical boundaries describe the payoff at the time of exercise for the option.

For example, a development or call option has boundary conditions as shown in figure 3. That is if the project is developed when the underlying asset price is P for a development cost of K , the payoff gives the boundary condition.

$$W(P | \text{development}) = \max\{0, P - K\}.$$

This is called the value-matching condition, as it says that when the option transitions from one state to another, (undeveloped to developed), the option values for the two states are equal. For a

14. This term is related to Jensen's inequality, which says that the expected value of a function of a random variable exceeds the function evaluated at the expected value of the random variable, if the function is convex.

European option, this payoff only occurs at the maturity date T , so we have

$$W(P_T, T) = \max\{0, P_T - K\}.$$

For an American option, we also need a condition to determine the boundary between the states where the option should be developed from those where it should be delayed. This is an optimization condition, and is sometimes called the smooth-pasting condition. It says that the option value, as a function of the underlying asset value, is tangent to the payoff function. The basis for this condition is apparent in figure 3, where we consider the solution of the PDE (10) subject to the value-matching condition and the management decision to develop the project as soon as the underlying asset value rises to some hurdle or trigger value P^\dagger . For the hurdle $P^\dagger = \$160$, for example, we see that the solution to the PDE dips below the payoff function for development, so it cannot be optimal. For example, the figure shows that the option value with this development policy is below the value that would be received if the option is developed at $P^* = \$266$. Thus, the best policy we can have is one where the option value function does not dip below the payoff at exercise. Hence it must be tangent.¹⁵

The smooth-pasting condition characterizes the optimal exercise or trigger point P^* for an arbitrary payoff function $\Pi(P, t)$:

$$W_p(P^*, t) = \Pi_p(P, t) \quad (11)$$

where

$$W_p(P, t) = \frac{\partial W}{\partial P}$$

and

$$\Pi_p(P, t) = \frac{\partial \Pi}{\partial P}.$$

Note that for a development option $\Pi_p(P, t) = \max\{0, P - K\}$, so in the region where there is a positive payoff (needed to justify early development), $\Pi_p = 1$.

15. Note that an option value function that is always above the payoff function is infeasible, because the option value must be on the payoff boundary when the option is exercised.

A. Analytic Solutions for Common Real Options

We can solve these partial differential equations and the appropriate boundary conditions to get solutions for some real options problems.

Black-Scholes Formula for European Call and Put Option on Lognormally Distributed Asset

We provide a general solution for a dividend yield δ . Let the call value be $W^C(P, t)$ and the put value be $W^P(P, t)$ for a european option expiring at time $T > t$, with exercise price K . Define

$$d_1 = \frac{\ln(P/K) + (r - \delta + \sigma_p^2/2)(T-t)}{\sigma_p \sqrt{T-t}}$$

$$d_2 = d_1 - \sigma_p \sqrt{T-t}.$$

Then

$$W^C(P, t) = Pe^{-\delta(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \quad (12)$$

$$W^P(P, t) = Ke^{-r(T-t)}N(-d_2) - Pe^{-\delta(T-t)}N(-d_1). \quad (13)$$

If we want to substitute $\alpha_p = r - \delta$, then we can rewrite these equations as:

$$d_1 = \frac{\ln(P/K) + (\alpha_p + \sigma_p^2/2)(T-t)}{\sigma_p \sqrt{T-t}}$$

$$W^C(P, t) = e^{-r(T-t)}(Pe^{\alpha_p(T-t)}N(d_1) - KN(d_2))$$

$$W^P(P, t) = e^{-r(T-t)}(KN(-d_2) - Pe^{\alpha_p(T-t)}N(-d_1)).$$

Perpetual American Call and Put Options on Lognormally Distributed Asset

The general solution to the valuation PDE (10) given the lognormal distribution parameters in equation (4) is

$$W(P) = A_1 P^{v_1} + A_2 P^{v_2}, \quad (14)$$

where v_1 is the positive root and v_2 is the negative root of

$$v = \frac{1}{2} - \frac{\alpha}{\sigma_P^2} \pm \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma_P^2}\right)^2 + \frac{2r}{\sigma_P^2}}. \quad (15)$$

For a development option, $A_2 = 0$ and for an abandonment option, $A_1 = 0$. Otherwise, A_i is determined by the payoff at the time the option is exercised and a smooth pasting optimality condition. First, ignore the smooth pasting condition and assume that the option is developed when the underlying asset price first hits some trigger price P^\dagger .

For the development (call) option, development occurs the first time that the underlying asset value P hits P^\dagger from below, and the value matching condition at exercise gives $A_1 = (P^\dagger - K)(P^\dagger)^{-v_1}$.

Thus, using the trigger development price P^\dagger , the perpetual american call option has the value

$$W^C(P, t) = (P^\dagger - K) \left(\frac{P}{P^\dagger}\right)^{v_1}. \quad (16)$$

But, we can choose the hurdle price P^\dagger to maximize A_1 , which maximizes the value of the call option. This is equivalent to the smooth pasting condition and is $P^\dagger = P^{C^*}$ where

$$P^{C^*} = \frac{v_1 K}{v_1 - 1}. \quad (17)$$

Similarly, for the abandonment (put) option, we have the optimal solution

$$W^P(P, t) = (K - P^{P^*}) \left(\frac{P}{P^{P^*}}\right)^{v_2}, \quad (18)$$

where

$$P^{P^*} = \frac{v_2 K}{v_2 - 1}. \quad (19)$$

VII. The Lattice or Tree Approach

The lattice approach to evaluating real options involves using a Bernoulli process with up and down jump moves at each step to approximate the stochastic process for the underlying “price” P . The jumps are spaced at time intervals of length h and a more precise approximation comes from setting h small. There are three parameters to the Bernoulli process:

- (1) The size of the up move, and its form (additive or multiplicative)
- (2) The size of the down move
- (3) The risk-neutral probability of the up move.

We choose these parameters to match the form and first two moments of the diffusion equation for the price. Recall equation (3):

$$dP = \alpha(P)dt + \sigma(P)d\omega.$$

where we can choose various functional forms and parameterizations for $\alpha(P)$ and $\sigma(P)$.

A. Geometric or Multiplicative Risk

If the diffusion term has multiplicative risk with volatility σ_P , or $\sigma(P) = \sigma_P P$, then we model the up and down move as being multiplicative. That is, if we start with price P_t , then we model the next price P_{t+h} as either taking the value uP_t or dP_t . By choosing

$$\begin{aligned} u &= e^{\sigma_P \sqrt{h}} \\ d &= e^{-\sigma_P \sqrt{h}}. \end{aligned} \tag{20}$$

we get a very good approximation to the diffusion term $\sigma(P)d\omega$ as h becomes small. Then, we choose the risk-neutral probability of an up move $\hat{\pi}$ so that the drift term $\hat{\alpha}(P)dt$ is well-approximated:

$$P + \hat{\alpha}(P)h = \hat{\pi}uP + (1 - \hat{\pi})dP$$

$$\pi(P) = \frac{\hat{\alpha}(P)h/P + 1 - d}{u - d}. \quad (21)$$

This is an adequate model in general and is just about the best we can get for the hybrid mean-reversion model.¹⁶ In the hybrid mean-reversion model, we have $\alpha(P)$, and hence $\hat{\alpha}(P)$ and $\hat{\pi}(P)$ varying throughout the tree.

For a lognormal diffusion, $\hat{\alpha}_p/P = \hat{\alpha}_p = r - \delta$, which suggests a risk-neutral probability of $(\hat{\alpha}_p + 1 - d)/(u - d)$. However, we can model the risk-neutral probability a little more precisely by noting that a risk-neutral investor would expect a growth factor of $e^{\hat{\alpha}_p h}$ over the time step, so that the risk-neutral probability is better written as

$$\hat{\pi} = \frac{e^{\hat{\alpha}_p h} - d}{u - d}. \quad (22)$$

These two representations of $\hat{\pi}$ converge as $h \rightarrow 0$.

B. Additive Risk Models

If the standard deviation in the diffusion term of equation (3) is constant $\sigma(P) = \sigma_p$, then we have an additive risk structure. We model the Bernoulli jumps as additive deviations, whereby P_{t+h} can take the value $P_t + U$ or $P_t + D$ where

$$\begin{aligned} U &= \sigma_p \sqrt{h} \\ D &= -\sigma_p \sqrt{h}. \end{aligned} \quad (23)$$

We use equation (9) to calculate the risk-neutral drift $\hat{\alpha}_p$, noting that the additive risk model makes the systematic risk measure constant: $\beta(P) = \beta_p$. The growth term $\alpha(P)$ takes the constant form $\alpha(P) = \alpha_p$ for the additive diffusion and the mean-reverting form $\alpha(P) = \lambda(\mu - P)$ for

16. Note that for hybrid mean reversion, the risk-neutral probability calculated in this manner might not be between 0 and 1. At extreme edges of the tree (only reached with low probability), the mean reversion to jump towards of the center of the tree may be so high that the probability is outside the interval [0, 1]. In general, it is satisfactory just to truncate it at 0 or 1.

the Ornstein-Uhlenbeck model.

We set the risk-neutral probability so that the Bernoulli process has the right risk-neutral drift over one step:

$$\begin{aligned}
 P + \hat{\alpha}(P)h &= \hat{\pi}(U + P) + (1 - \hat{\pi})(D + P) \\
 \hat{\pi}(P) &= \frac{\hat{\alpha}(P)h - D}{U - D} \\
 &= \frac{(\alpha(P) - \gamma\beta_0)h - D}{U - D}. \tag{24}
 \end{aligned}$$

C. Recursive Computation of Option Values on a Lattice

Having specified the dynamics of the Bernoulli model in on step of the lattice, we can combine them into a whole lattice of up and down moves. This gives us a specialized version of the decision tree because the branches of the tree recombine. That is, at a given point in time, we can fully describe the state by knowing the total numbers of up jumps and down jumps — we do not need to know the order of up and down jumps. Thus, an up-jump followed by a down-jump leads to the same state as a down-jump followed by an up-jump.

To compute option values and strategy, we take the standard notion of a decision tree whereby we start at the tips of the tree, which correspond to the terminal date of the option and move back one step earlier in time time by computing the continuation value of the option as:

$$e^{-rh} (\hat{\pi}(P)W(P_u) + (1 - \hat{\pi}(P))W(P_d)). \tag{25}$$

Here, the values P_u and P_d correspond to the up and down jump values starting at price P as given in either equation (20) or (23), for multiplicative and additive processes, respectively. The continuation value is the value of the option if it is continued in the same operating state (exercised or unexercised) at the end of the period as at the beginning.

We then compare the continuation value to the payoff that corresponds to exercising the option at that point and assign the larger of these to the option value $W(P)$. This optimization is known as the

Bellman equation or the principal of optimality in dynamic programming. The optimum will satisfy the smooth pasting condition that characterizes optimal exercise, but we do not need to compute any partial derivatives in this analysis. Implementing the smooth pasting condition directly is usually quite difficult, but the Bellman equation is easy to implement.

The payoff can be the payoff to develop $\max\{0, P - K\}$ or abandon $\max\{0, K - P\}$, or any general payoff that is contingent on the underlying price P , bearing in mind that P could represent a quantity of goods, degree days of heat, etc.

To summarize, we calculate the option value recursively to earlier points in the tree with the principle of optimality or Bellman equation:

$$W(P) = \max\left\{\text{Exercise Payoff}(P), e^{-rh}(\pi(P)W(P_u) + (1 - \pi(P))W(P_d))\right\}. \quad (26)$$

VIII. The Least-Squares Monte Carlo Approach

The Bellman equation (26) recursively computes the optimal real option value by comparing the continuation value as in equation (25) to the proceeds of switching to the next state (exercising the option in our example). The continuation value is the present value of the risk-neutral expected payoff from continuing for a further period in the same mode (e.g. undeveloped). The Bellman equation determines value as well as optimal strategy or operating policy. The lattice method is simply one technique for computing estimates of the risk-neutral expected value of continuing in the same mode and switching modes, each conditional on being in a given point in the lattice. This suggests that if we can find any other way of estimating conditional expectations, we can still perform the Bellman comparison and determine option value and strategy.

Moreover, our discussion in subsection III, A, shows that we can accurately estimate the optimal option value even if we make fairly large errors in the specification of the precise trigger boundary. If we use these approximately correct trigger strategies with a valuation technique that is quite accurate given these trigger strategies, then we can still accurately estimate option value and determine strategies that are good enough to achieve value close to the optimum.

These observations enable us to employ the least-squares Monte

Marlo (LSM) approach. The idea is to estimate the conditional expected continuation value from a simulation of the whole distribution, instead of using just a Bernoulli lattice.¹⁷ By noting the optimal decisions, we can map out the trigger boundary for the transition decisions. Then, we can go back to the original simulation to compute the present value of the (risk-neutral) expected payoff arising from the trigger strategy along each sample path. Even if we don't compute the trigger boundary exactly right, we will be able to compute the real option value quite accurately, and we will have a description of a trigger strategy that achieves a value close to the optimum.

The traditional Monte Carlo simulation for option valuation (first introduced by Boyle (1977)) is a forward-looking technique, whereas dynamic programming implies backward recursion. Many approaches have been recently proposed to properly match simulation and dynamic programming: these include Bossaerts (1989), Cortazar and Schwartz (1998) and Broadie and Glasserman (1997).

We illustrate the Longstaff and Schwartz (2001) least squares Monte Carlo (LSM) method. It is based on a Monte Carlo simulation and uses least squares regression to estimate the continuation value of the Bellman equation and hence the optimal policy of the problem. Gamba (2003) extends the LSM approach to the real options problem with many interacting real options many state variables, including optimal switching problems.

In what follows we provide an introduction of the LSM approach for the real options valuation the option to switch modes of operation.

A. A Monte Carlo valuation of the option to switch operating mode

Consider a switching problem of opening and closing a plant as described in Brennan and Schwartz (1985) and Dixit (1989b). The plant produces a commodity that has a stochastic spot price P , which follows the lognormal diffusion (4) with risk-neutral drift rate $\hat{\alpha}$ as in (22). There are two operating modes: closed $z = c$, with no production, but a maintenance cost of m per year, and open $z = o$, with a production rate q per year and unit operating cost C . Production can be suspended (switching from o to c) at a switching cost $S_{o,c}$, or it can be restarted

17. In section VII we estimated the continuation value using the risk-neutral probability π . With the LSM, we will estimate the continuation value using a regression or least-squares projection. In each case, we want to estimate the continuation value as the present value of the expectation of continuing in a given state.

(switching from c to o) at a switching cost $S_{c,o}$.

Consider the operation of the facility over the time span $[0, T]$, and divide it into N intervals of length $h = T/N$. The dynamics of the state variable are simulated by generating K paths for P . The operating cash flow for one time step when the commodity price is P , the starting mode z and the ending mode ζ , is $X(P, z, \zeta)$ where

$$\begin{aligned} X(P, o, o) &= q(P - C)h \\ X(P, c, c) &= -mh \\ X(P, o, c) &= -mh - S_{o,c} \end{aligned} \tag{27}$$

$$X(P, c, o) = q(P - C)h - S_{c,o}.$$

Starting at the operating mode $z \in \{o, c\}$, let the value of an operating facility be $W(P, t, z)$, and the optimal strategy be $\psi(P, t, z) \in \{o, z\}$. We recursively compute these values in a backwards fashion from $t = T$ to $T - h, T - 2h, \dots, 0$ by an extension of the Bellman equation (26):

$$W(P, t, z) = \max_{\zeta \in \{o, c\}} \left(X(P, z, \zeta) + e^{-rh} \hat{E}_t [W(P_{t+h}, t+h, \zeta)] \right) \tag{28}$$

$$\psi(P, t, z) = \arg \max_{\zeta \in \{o, c\}} \left(X(P, z, \zeta) + e^{-rh} \hat{E}_t [W(P_{t+h}, t+h, \zeta)] \right). \tag{29}$$

For the optimal strategy in (29), we take the convention that the mode is unchanged if the two maximands have the same value. Here, the expectation with respect to the the risk-neutral distribution $\hat{E}[\bullet]$ generalizes the risk-neutral probability $\hat{\pi}$ in the lattice approach and corresponds to the dynamics of P with the drift rate $\hat{\alpha}$.

To this point, we haven't proposed anything new. The special insight is to replace the one-period conditional estimation $\hat{E}_t[\bullet]$ in equations (28) and (29) with approximations based on a least squares projection of the time $t + h$ values on information that is known at time t , such as polynomials in P_t . These will be good enough estimates of the conditional risk-neutral expectation to form a good strategy $\psi(P, t, z)$. There are many bases besides polynomials that we can use, but here we will use the polynomials $1, P, P^2$ and P^3 .

We implement this least-squares projection at each point in time t , using a backwards recursion from $t = T$. Suppose we have the optimal strategy policy $\psi(P_\tau, \tau, z)$ for $\tau = T, T - h, \dots, t + h$, any $z \in \{o, c\}$ and any P_τ along a path on the simulation. Then we can estimate the time- t expectation of next-period plant value $W(P_{t+h}, t + h, \zeta)$ as a least-squares projection onto the polynomial basis $\{1, P_t, P_t^2, P_t^3\}$ of the present value of all cash flows along the sample paths through P_{t+h} , given that the plant is in operating mode ζ at time $t + h$. That is, for each simulation path ω that goes through P_{t+h} , we have a commodity price sequence $P_{t+h}, P_{t+2h}, \dots, P_T$. Also, going forward recursively from time $t + h$ in operating mode ζ , we can use the strategy ψ to determine the sequence of operating modes for times $t + h, t + 2h, \dots, T$ along the simulation path ω :

<u>Time</u>	<u>Commodity Price</u>	<u>Strategy</u>
$t + h$	P_{t+h}	ζ
$t + 2h$	P_{t+2h}	$\psi(P_{t+2h}, t + 2h, \zeta)$
$t + 3h$	P_{t+3h}	$\psi(P_{t+3h}, t + 3h, \psi(P_{t+2h}, t + 2h, \zeta))$
\vdots	\vdots	\vdots

Thus, along each path ω , we can determine the sequence of cash flows $X(P_{t+h}, z, \psi(P_t, t, z)), \dots$ for times $t + h, t + 2h, \dots, T$. Denote the present value, discounted to time t , of this stream of cash flows by $\Phi(t, \zeta, \omega)$. Now, this value is not known at time t because it uses information about the path beyond time t . But, we can estimate its conditional expected value by a least squares regression of all of these values on some explanatory functions of the observable P_t , say the polynomials 1, P , P^2 , and P^3 . That is, for a given time t , consider selecting a strategy starting in operating mode ζ for time $t + h$. We can consider a population of all possible paths ω and use as explanatory variables the polynomials in $P = P_t$, where the random variable P depends on t and ω . The dependent variable is the present value $\Phi(t, \zeta, \omega)$ of the payoffs obtained when starting from this mode ζ at time t :

$$\Phi(t, \zeta, \omega) = \phi_0(t, \zeta) + \phi_1(t, \zeta)P + \phi_2(t, \zeta)P^2 + \phi_3(t, \zeta)P^3 + \varepsilon_{t, \zeta, \omega}. \quad (30)$$

Given the values of these regression coefficients $\phi_i(t, \zeta)$, we can form a

conditional estimate of the option continuation values for strategies $\zeta \in \{o, c\}$, discounted to time t , which we will denote $e^{-rh} \tilde{E}_t[W]$:

$$e^{-rh} \tilde{E}_t[W(P_{t+h}, t+h, \zeta)] = \phi_0(t, \zeta) + \phi_1(t, \zeta) P_t + \phi_2(t, \zeta) P_t^2 + \phi_3(t, \zeta) P_t^3. \quad (31)$$

We will substitute this expected value into the Bellman equation (29) to get the approximately optimal strategy ψ :

$$\psi(P, t, z) = \arg \max_{\zeta \in \{o, c\}} (X(P, z, \zeta) + e^{-rh} \tilde{E}_t[W(P_{t+h}, t+h, \zeta)]). \quad (32)$$

This fully describes the approximation of the optimal operating strategy. One might think that it is also a good idea to use $\tilde{E}_t[W(P_{t+h}, t+h, \zeta)]$ in equation (28) in order to recursively compute real option values back to $t = 0$. However, each of these estimates $\tilde{E}_t[W]$ of option value are computed with error, and using them in the Bellman equation with the maximization amounts to taking a convex function of them. By Jensen's inequality, this will result in an over-estimate of true real option value W . To avoid this bias, we can simply compute the option value W as the average over all simulation paths of the present value of the payoffs Φ . That is

$$W(P, 0, z) = e^{-rh} \frac{1}{K} \sum_{\omega=1}^K (X(P, z, z) + \Phi(0, z, \omega)). \quad (33)$$

This gives a starting value for plants initially in each of the open and the closed states.

B. Example of the Monte Carlo Method

Consider this model with parameters $r = 4\%$, $T = 4$, $q = 10$, $m = -1$, $S_{c,o} = 8$, $S_{o,c} = 4$ and $C = 0.8$. $\sigma_p = 35\%$, $\delta = 0.03$, so that $\alpha = 0.04 - 0.03 = 1\%$. We take $K = 16$ paths and $N = 4$ time steps so that $h = T/N = 1$. Table 1 shows the simulated sample paths for the price P .

We apply backward induction to solve this problem. At $n = 4$ ($t = T = 4$), the payoff of the open facility is

TABLE 1. Commodity product price dynamics P (16 paths and 4 time steps), with initial price $P_0 = 1$.

Path	t			
	1	2	3	4
1	0.9538	1.3495	1.0589	1.1922
2	0.7580	0.5668	0.4004	0.3088
3	1.2596	0.7613	0.6636	0.5982
4	1.0303	0.9541	1.1432	0.9328
5	0.6719	0.5686	0.4006	0.3701
6	1.5183	1.0737	0.6699	1.0892
7	1.0514	1.1889	1.0831	0.8322
8	1.5942	2.5500	2.3679	1.4038
9	0.9463	0.6037	0.6944	0.5566
10	1.1907	1.4373	1.8365	2.1491
11	0.7166	1.0700	1.1081	1.1093
12	0.8761	0.8538	0.6432	0.7115
13	1.3433	1.4328	1.8356	1.7929
14	0.5945	0.7587	1.0976	0.6093
15	0.8585	0.6852	0.6789	0.7975
16	0.5662	0.3195	0.3105	0.4728

$$\begin{aligned}
X(P_4, o, \psi) &= \max\{X(P_4, o, o), X(P_4, o, c)\} \\
&= \max\{q(P_4 - C)h, -mh - S_{o,c}\},
\end{aligned}$$

whereas the payoff of the closed facility is

$$\begin{aligned}
X(P_4, c, \psi) &= \max\{X(P_4, c, o), X(P_4, c, c)\} \\
&= \max\{q(P_4 - C)h, -S_{c,o} - mh\}.
\end{aligned}$$

These payoffs are shown in the 6th columns of the upper (initially open) and lower (initially closed) panels of table 2. The path-wise optimal discounted values discounted to $t = 3$, $\Phi(3, \zeta, \omega)$ are shown in the 5th columns. The OLS regressions (30) of $\Phi(3, \zeta, \omega)$ on the polynomial in P are shown below the panels for open and closed, respectively. From equation (31), we use these OLS regressions to get the approximate discounted expected continuation values $e^{-r} \tilde{E}_3[W(P_4, 4, \zeta)]$, $\zeta \in \{o, c\}$, as shown in the 4th columns of each panel.

TABLE 2. Computations at $t = 3$.

Open at start of $t = 3, z = o$					
Path	Bellman Optimality		$e^{-th} \tilde{E}_3 [W]$	Φ	$X(P_t, o, \psi)$
	$\zeta = o$	$\zeta = c$			$t = 4$
1	4.639	-5.529	2.050	3.76	3.922
2	-6.974	-5.784	-2.977	-4.719	-4.912
3	-3.370	-6.644	-2.006	-1.939	-2.018
4	6.537	-5.083	3.105	1.276	1.328
5	-6.972	-5.785	-2.977	-4.130	-4.299
6	-3.261	-6.646	-1.960	2.778	2.892
7	5.181	-5.405	2.350	0.309	0.322
8	22.299	-5.534	6.620	5.801	6.038
9	-2.834	-6.643	-1.778	-2.338	-2.434
10	20.204	-2.160	9.839	12.962	13.491
11	5.744	-5.273	2.663	2.972	3.094
12	-3.713	-6.635	-2.145	-0.851	-0.885
13	20.191	-2.160	9.835	9.540	9.930
14	5.507	-5.329	2.531	-1.832	-1.907
15	-3.106	-6.646	-1.895	-0.024	-0.025
16	-7.720	-5.065	-2.825	-3.144	-3.272

$e^{-th} \tilde{E}_3 [W(P_4, 4, o)] = 0.1852 - 16.7757 P + 25.0070 P^2 - 7.0842 P^3$

Closed at start of $t = 3, z = c$					
Path	Bellman Optimality		$e^{-th} \tilde{E}_3 [W]$	Φ	$X(P_t, c, \psi)$
	$\zeta = o$	$\zeta = c$			$t = 4$
1	-3.361	-1.529	-0.529	-0.961	-1.000
2	-14.974	-1.784	-0.784	-0.961	-1.000
3	-11.370	-2.644	-1.644	-0.96	-1.000
4	-1.463	-1.083	-0.083	-0.961	-1.000
5	-14.972	-1.785	-0.785	-0.961	-1.000
6	-11.261	-2.646	-1.646	-0.961	-1.000
7	-2.819	-1.405	-0.405	-0.961	-1.000
8	14.299	-1.534	-0.534	-0.961	-1.000
9	-10.834	-2.643	-1.643	-0.961	-1.000
10	12.204	1.840	2.840	5.275	5.491
11	-2.256	-1.273	-0.273	-0.961	-1.000
12	-11.713	-2.635	-1.635	-0.961	-1.000
13	12.191	1.840	2.840	1.854	1.930
14	-2.493	-1.329	-0.329	-0.961	-1.000
15	-11.106	-2.646	-1.646	-0.961	-1.000
16	-15.720	-1.065	-0.065	-0.961	-1.000

$e^{-th} \tilde{E}_3 [W(P_4, 4, c)] = 4.5418 - 20.8286 P + 21.0184 P^2 - 5.5440 P^3$

TABLE 3. Computations at $t = 2$.

Open at start of $t = 2, z = o$						
Path	Bellman Optimality		$e^{-th} \tilde{E}_2 [W]$	Φ	$X(P_t, o, \psi)$	
	$\zeta = o$	$\zeta = c$			$t = 3$	$t = 4$
1	18.679	0.600	13.183	6.108	2.589	3.922
2	-7.269	-7.981	-4.937	-5.727	-5.000	-1.000
3	-2.879	-8.156	-2.492	-3.174	-1.364	-2.018
4	3.252	-6.452	1.711	4.524	3.432	1.328
5	-7.239	-7.993	-4.924	-5.727	-5.000	-1.000
6	7.664	-4.695	4.927	1.420	-1.301	2.892
7	12.189	-2.639	8.299	3.017	2.831	0.322
8	38.361	8.249	20.861	20.637	15.679	6.038
9	-6.605	-8.186	-4.642	-3.261	-1.056	-2.434
10	22.199	2.453	15.826	22.412	10.365	13.491
11	7.524	-4.755	4.824	5.816	3.081	3.094
12	-0.120	-7.538	-0.658	-2.324	-1.568	-0.885
13	22.021	2.358	15.693	19.116	10.356	9.930
14	-2.950	-8.167	-2.538	1.099	2.976	-1.907
15	-4.848	-8.344	-3.700	-1.187	-1.211	-0.025
16	-9.219	-4.141	-4.413	-5.727	-5.000	-1.000

$e^{-th} \tilde{E}_2 [W(P_3, 3, o)] = 4.3169 - 44.2143 P + 57.5835 P^2 - 14.7845 P^3$

Closed at start of $t = 2, z = c$						
Path	Bellman Optimality		$e^{-th} \tilde{E}_2 [W]$	Φ	$X(P_t, c, \psi)$	
	$\zeta = o$	$\zeta = c$			$t = 3$	$t = 4$
1	10.679	4.600	5.600	-1.884	-1.000	-1.000
2	-15.269	-3.981	-2.981	-1.884	-1.000	-1.000
3	-10.879	-4.156	-3.156	-1.884	-1.000	-1.000
4	-4.748	-2.452	-1.452	-1.884	-1.000	-1.000
5	-15.239	-3.993	-2.993	-1.884	-1.000	-1.000
6	-0.336	-0.695	0.305	-1.884	-1.000	-1.000
7	4.189	1.362	2.362	-1.884	-1.000	-1.000
8	30.361	12.249	13.249	12.951	7.679	6.038
9	-14.605	-4.186	-3.186	-1.884	-1.000	-1.000
10	14.199	6.453	7.453	14.726	2.365	13.491
11	-0.476	-0.755	0.245	-1.884	-1.000	-1.000
12	-8.120	-3.538	-2.538	-1.884	-1.000	-1.000
13	14.021	6.358	7.358	11.429	2.356	9.930
14	-10.950	-4.167	-3.167	-1.884	-1.000	-1.000
15	-12.848	-4.344	-3.344	-1.884	-1.000	-1.000
16	-17.219	-0.141	0.859	-1.884	-1.000	-1.000

$e^{-th} \tilde{E}_2 [W(P_3, 3, c)] = 13.5465 - 55.6235 P + 53.8132 P^2 - 12.567 P^3 + \epsilon$

TABLE 4. Computations at $t = 1$.

Open at start of $t = 1, z = o$							
Path	Bellman Optimality		$e^{-th} \tilde{E}_1 [W]$	Φ	$X(P_t, o, \psi)$		
	$\zeta = o$	$\zeta = c$			$t = 2$	$t = 3$	$t = 4$
1	4.838	-4.988	3.300	11.149	5.495	2.589	3.922
2	-1.625	-7.084	-1.205	-7.743	-2.332	-5.000	-1.000
3	17.058	1.275	12.462	-3.421	-0.387	-1.364	-2.018
4	7.659	-3.777	5.356	5.827	1.541	3.432	1.328
5	-4.110	-7.572	-2.829	-7.726	-2.314	-5.000	-1.000
6	29.253	9.737	22.070	3.993	2.737	-1.301	2.892
7	8.465	-3.403	5.951	6.636	3.889	2.831	0.322
8	33.127	12.813	25.185	36.642	17.500	15.679	6.038
9	4.571	-5.095	3.108	-5.020	-1.963	-1.056	-2.434
10	14.091	-0.473	10.184	27.656	6.373	10.365	13.491
11	-2.848	-7.350	-2.014	8.182	2.700	3.081	3.094
12	2.140	-5.989	1.379	-1.716	0.538	-1.568	-0.885
13	20.830	3.682	15.396	24.446	6.328	10.356	9.930
14	-6.150	-7.798	-4.095	0.660	-0.413	2.976	-1.907
15	1.554	-6.184	0.969	-2.244	-1.148	-1.211	-0.025
16	-6.849	-7.831	-4.511	-6.614	-5.000	-1.000	-1.000

$e^{-th} \tilde{E}_1 [W(P_2, 2, o)] = -7.2379 - 5.1135 P + 18.4058 P^2 - 1.5311 P^3$

Closed at start of $t = 1, z = c$							
Path	Bellman Optimality		$e^{-th} \tilde{E}_1 [W]$	Φ	$X(P_t, c, \psi)$		
	$\zeta = o$	$\zeta = c$			$t = 2$	$t = 3$	$t = 4$
1	-3.162	-0.988	0.012	3.462	-2.505	2.589	3.922
2	-9.625	-3.084	-2.084	-2.771	-1.000	-1.000	-1.000
3	9.058	5.275	6.275	-2.771	-1.000	-1.000	-1.000
4	-0.341	0.223	1.223	-2.771	-1.000	-1.000	-1.000
5	-12.110	-3.572	-2.572	-2.771	-1.000	-1.000	-1.000
6	21.253	13.737	14.737	-3.693	-5.263	-1.301	2.892
7	0.465	0.597	1.597	-1.051	-4.111	2.831	0.322
8	25.127	16.813	17.813	28.956	9.500	15.679	6.038
9	-3.429	-1.095	-0.095	-2.771	-1.000	-1.000	-1.000
10	6.091	3.527	4.527	19.970	-1.627	10.365	13.491
11	-10.848	-3.350	-2.350	0.496	-5.300	3.081	3.094
12	-5.860	-1.989	-0.989	-2.771	-1.000	-1.000	-1.000
13	12.830	7.682	8.682	16.760	-1.672	10.356	9.930
14	-14.150	-3.798	-2.798	-2.771	-1.000	-1.000	-1.000
15	-6.446	-2.184	-1.184	-2.771	-1.000	-1.000	-1.000
16	-14.849	-3.831	-2.831	-2.771	-1.000	-1.000	-1.000

$e^{-th} \tilde{E}_1 [W(P_2, 2, c)] = 1.3476 - 14.5983 P + 11.1579 P^2 + 2.8089 P^3$

TABLE 5. Computations at $t = 0$.

Open at start of $t = 1, z = o$					
Path	Φ	$X(P_t, o, \psi)$			
		$t = 1$	$t = 2$	$t = 3$	$t = 4$
1	12.189	1.538	5.495	2.589	3.922
2	-7.843	-0.420	-2.332	-5.000	-1.000
3	1.129	4.596	-0.387	-1.364	-2.018
4	7.811	2.303	1.541	3.432	1.328
5	-8.654	-1.281	-2.314	-5.000	-1.000
6	10.738	7.183	2.737	-1.301	2.892
7	8.791	2.514	3.889	2.831	0.322
8	42.836	7.942	17.500	15.679	6.038
9	-3.417	1.463	-1.963	-1.056	-2.434
10	30.326	3.907	6.373	10.365	13.491
11	7.060	-0.834	2.700	3.081	3.094
12	-0.918	0.761	0.538	-1.568	-0.885
13	28.708	5.434	6.328	10.356	9.930
14	-1.341	-2.055	-0.413	2.976	-1.907
15	-1.594	0.585	-1.148	-1.211	-0.025
16	-8.601	-2.338	-5.000	-1.000	-1.000
Closed at start of $t = 1, z = c$					
Path	Φ	$X(P_t, c, \psi)$			
		$t = 1$	$t = 2$	$t = 3$	$t = 4$
1	2.366	-1.000	-2.505	2.589	3.922
2	-3.623	-1.000	-1.000	-1.000	-1.000
3	-6.557	-3.405	-0.387	-1.364	-2.018
4	-3.623	-1.000	-1.000	-1.000	-1.000
5	-3.623	-1.000	-1.000	-1.000	-1.000
6	3.052	-0.817	2.737	-1.301	2.892
7	-1.970	-1.000	-4.111	2.831	0.322
8	35.150	-0.058	17.500	15.679	6.038
9	-3.623	-1.000	-1.000	-1.000	-1.000
10	22.639	-4.093	6.373	10.365	13.491
11	-0.485	-1.000	-5.300	3.081	3.094
12	-3.623	-1.000	-1.000	-1.000	-1.000
13	21.021	-2.567	6.328	10.356	9.930
14	-3.623	-1.000	-1.000	-1.000	-1.000
15	-3.623	-1.000	-1.000	-1.000	-1.000
16	-3.623	-1.000	-1.000	-1.000	-1.000

Next, the optimal decision at $t = n = 3$ is made by equation (32). The two strategy choices are $\zeta = o$ and $\zeta = c$. The values of these strategies differ, depending on whether the starting mode is $z = o$ (top panel) or z

$= c$ (bottom panel). Thus, the 2nd column in the top panel describes the payoff to the strategy of starting in the open state at $n = 3$ and remaining in that state. These are the column-4 values plus the payoff for $n = 3$, which is $X(P_3, o, o)$ in the open-open case, for example. The cash flow includes any transition costs, as in equation (27).

By comparing the values for the two strategies, the optimal policy is determined and the new optimal payoffs, $X(P, z, \psi)$, are then computed for decisions in the time period $n = 2$, and placed in the 6th and 7th columns of the open and closed panels of table 3. The streams of payoffs come from the transitions in the strategy ψ and the initial state (open or closed), price P and time t .¹⁸ Once again, by discounting, we compute $\Phi(2, \zeta, \omega)$ for $\zeta \in \{o, c\}$ and then, by OLS, we estimate the continuation values $e^{-r} \tilde{E}_2[W(P_2, 2, \zeta)]$, $\zeta \in \{o, c\}$. These are used to calculate the values for the Bellman optimality comparison in the first two columns of table 3. The resulting optimal cash flows are carried to the analysis at $n = 1$ in the last three columns of table 4.

At $n = 1$, the above computations are repeated, giving the optimal payout streams $X(P_1, o, \psi), \dots$ and $X(P_1, c, \psi), \dots$ for each path ω . These values are placed in table 5. The present values of these payouts are the $\Phi(0, z, \omega)$ in the first column of each panel of the table.

Then, the values of the open and closed facility are obtained by taking the sample average over all ω of $\Phi(0, z, \omega)$ for $z \in \{o, c\}$. The estimated value of an open facility at $t = 0$ is $W(1, 0, o) = 7.236$, and the value of a closed facility is $W(1, 0, c) = 2.890$.

A more accurate valuation of the facility can be obtained by increasing the number of paths and the number of steps.¹⁹

IX. Concluding Remarks

We have given an overview of several issues that we find are important to real option analysis. Although real options analysis is intellectually appealing and rigorous, we continue to be surprised that it has not achieved the popularity of other techniques in finance. For example, corporations invest significant resources in managing financial risk, not

18. Note that it can happen that the facility is kept in operation with a negative value, because we have not explicitly considered the option to abandon.

19. Additional details are given in Gamba (2003).

just in options and insurance premia, but in the organizational infrastructure and labour cost needed to manage a derivatives, futures and swaps portfolio. They also expend many resources to compute risk measures such as Value at Risk (VaR), even when they are not required to do this — financial institutions are required to calculate VaR, but other organizations do it voluntarily.

These expensive corporate activities are designed to model and measure risk. But, only real options strategy can go the extra step and use this risk to create shareholder value. This is why we are so perplexed as to the lack of broad endorsement of real options strategies. The first part of this paper identifies some of the important characteristics of real options that are different from the existing popular financial risk analysis techniques. Some of these are often overlooked. For example, we have seen people try to build real options strategies without flexibility. Others don't realize how leverage allows real options to create value and also prevents analysis by risk-adjusted discount rates.

We also discussed organizational impediments to adoption of real options strategy. With a greater awareness of these problems and advantages of real options, proponents can more successfully get a value-enhancing real options strategy adopted in an organization.

The second half of the paper is more analytical, but is designed to present a basic shell of techniques that are valuable for the real options analyst. They include models of flexibility: decision trees and influence diagrams. They also include models of risk: continuous-time (diffusion) and discrete (lattice) models of risk. Issues associated with setting parameters for these models have been discussed, as well as reasons why some models like mean reversion are important in various settings.

The paper closes with discussions of two numerical techniques that can be used to solve a broad range of real options problems: the lattice or tree approach and the least-squares Monte Carlo method.

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