We consider the problem of deleveraging a large long-short portfolio of risky assets in a relatively short trading period to satisfy specific leverage and margin policy constraints in the presence of liquidity impact on prices. Given the future uncertainty during the rebalancing period, the model is formulated to determine the impact of policy limits on mean-variance tradeoff frontier. Liquidity costs are considered due to both quantity and intensity of trading leading to temporary and permanent impact on asset prices. Utilizing a separable upper approximating risk metric on portfolio variance, we formulate the model as a quadratic separable, but nonconvex, optimization problem, which is extremely-difficult to solve using known techniques. We develop, first, a new dual cutting plane methodology to solve the non-convex Lagrangian dual problem efficiently. Starting from the Lagrangian solution, next, we develop a method to obtain a sequence of progressively-improving feasible deleveraged portfolios. Using real data on ETF assets, the methodology is empirically-tested, and the optimal deleveraging strategy sensitivity to leverage and margin limits is analyzed in order to develop managerial insight for setting policy parameters. Relative gain from our model in comparison to ignoring market liquidity is quantified using out-of-sample analysis.

Key words: Portfolio deleveraging, market impact of trading, Lagrangian dual cutting planes, nonconvex quadratic separable optimization.

1. Introduction

As evidenced by historical market crashes, leveraged and margin-enabled portfolios of high NAV encounter increased pressure to reduce portfolio risk exposure in adverse market situations. For example, in the crash of 2009, Lehman Brothers portfolio with a leverage ratio of over 31:1 failed to raise sufficient capital to sustain the portfolio losses brought on by the rapidly falling asset prices under extreme pressures of illiquidity, leading to the company’s demise. Likewise, during the financial crisis of 1998, Long-Term Capital Management which had once leveraged their equity as much as 30:1 lost a staggering 44% of its equity during the month of August due to massive losses.
under forced liquidation of the holdings to cover margin requirements (Jorion 2000). The asset management industry is fraught with many such historical events associating financial ruin and high leverage during market turbulence. Hence, prudent portfolio deleveraging and management practices are indispensable under impending market downturns.

Using leverage (i.e. borrowing on margin for asset purchase) is often a mechanism used to realize the full growth potential of an investment strategy. While the preceding historical artifacts have led to the connotation that leverage is dangerous, leverage in itself is not the main cause for alarm, but it is the underlying portfolio’s (volatility) risk which indeed is amplified under leverage. That is, leveraging a high-risk unlevered portfolio carries much danger relative to a low-risk portfolio being leveraged to the same degree to improve target return. In addition, sufficient collateral should be available through long asset/cash positions to support portfolio short positions and avoid forced liquidation.

As market conditions evolve, a well-structured portfolio at the present time may become poorly-structured with regard to the above concerns in future adverse market scenarios, accompanied with elevated volatilities. Then, it is imperative that the portfolio is deleveraged to satisfy the borrowing policy and restructured for an acceptable risk-return profile whilst achieving a revised long-short balance under the specified margin restrictions. For funds with significant NAV, such a rebalancing operation also encounters elevated asset trading illiquidities during market downturn scenarios, thus, exacerbating the negative portfolio effects. As such, the portfolio deleveraging and rebalancing process must consider risk-return trade-off, leverage and margin limits, along with liquidity impact in trading ex-ante when determining optimal asset positions. This is the premise of the model and analysis presented in this paper.

Under Modern Portfolio Theory (MPT), see Markowitz (1952), MV optimal portfolios may be endowed with long-short leverage levels inconsistent with the investor’s leverage aversion, especially at higher portfolio volatility levels. In extending the MPT, Jacobs and Levy (2012, 2013, 2014) proposed to augment the mean-variance utility with a leverage risk averse component so that optimal portfolios so-determined are representative of the investor’s leverage preference. This is tantamount to indirect control of long-short exposures to satisfy margin requirements. As Asness et al. (2012) points out portfolios of safer assets combined in low levels of long-short leveraging may be further invested under external borrowing to improve returns, the latter referred to as financial leveraging. This indeed is the rationale behind the so-called risk parity portfolios which allocates portfolio risk equally among different asset classes, and under financial-leveraging those less risky assets are further invested upon, i.e., overweighting safer assets. However, we note that

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1 Within the investment industry, the hedge fund sector makes the most use and employs high levels of leverage (Ang et al. 2011).
these discussions have not incorporated liquidity impact, and that even the risk parity portfolios may end up overly-leveraged and in violation of margin policies when markets evolve under negative scenarios. The work in our paper is applicable for deleveraging such portfolios to rid them of excessive borrowing in the risky and risk-free assets in those markets punctuated with substantial illiquidity within the framework of MV utility.

Our work may also be viewed as a generalization of the model by Brown et al. (2010) who developed a deleveraging model that maximizes net equity of a long-only portfolio while satisfying a target leverage ratio. Although their model incorporates trading impact via temporary and permanent impact on asset prices based on the quantity and intensity of trading, it is free of long-short leveraging or any portfolio risk. Price impact is only based on asset sales for deleveraging long portfolios. Moreover, they assume a condition on the price impact parameters so that the resulting optimization model is guaranteed to be a convex, separable quadratic problem with a single constraint, solvable with relative ease. Chen et al. (2014) relaxes the latter assumption and develop an algorithm to solve the resulting indefinite separable singly-constrained quadratic program. Chen et al. (2015) extend the model for the case when temporary price impact is a higher-order power function of the trading rate.

The deleveraging models in the foregoing studies assume that the initial portfolio holds only long positions, and thus, deleveraging is to sell of assets to satisfy a financial leverage constraint. Our work relaxes this assumption and allows portfolio long and short positions; this implies that deleveraging involves not only selling, but also cover-buying. Our deleveraging strategy is limited to partial sale in long positions and purchases in short positions (partial cover buying). That is, in the deleveraged portfolio, neither the initial long positions become short nor initial short positions become long. Moreover, collateral using marginable long asset positions must be available to cover some or all of the credit risk that the short portfolio poses for the counterparty. The allowable margin could be a different percentage for each asset, and in order to avoid margin call if the balance available falls below the amount allowed, the short portfolio is subject to margin limits (Brumm et al. 2015). Consequently, the problem in this paper becomes a multi-constrained, nonconvex, mean-risk optimization model.

The second important contribution is that we consider the deleveraging problem in a utility framework. In the earlier work mentioned above, deleveraging (within a short time window) is performed to maximize the total asset value at the end of the deleveraging period without considerations of portfolio risk resulting from market evolution in the ensuing period. For instance, a portfolio deleveraged over a day would still be held over a longer period (e.g., several weeks), resulting in unacceptable portfolio risk due to improperly-chosen assets and positions so-reduced. Portfolios deleveraged without any account of risk is contrary to the safe-betting concept in risk
parity. Consequently, in our deleveraging model both long and short asset positions are shrunk simultaneously under liquidity costs to meet specified leverage and margin limits whilst achieving an efficient deleveraged portfolio in a mean-risk framework considering the portfolio performance outside the deleveraging period.

Also, significantly, we focus on developing guidance on setting leverage/margin policy parameters judiciously as market uncertainties evolve so that financial turmoil for the institution or the fund can be avoided. This requires developing insights on the trade-off frontier that results from policy parameters and portfolio risk-return characteristics. This necessitates solving the proposed deleveraging optimization model repeatedly given that the model is indefinite, nonseparable quadratic, and multi-constrained. Solving optimization models with an indefinite objective and multiple indefinite constraints even once is notoriously-difficult, especially when the number of variables (i.e. assets) is large, and thus, we develop a novel and efficient solution methodology, another significant contribution in this paper.

The solution methodology we develop in this paper is based on three steps. First, we exploit a separable correlation-free upper bounding construct for the portfolio variance, under which the model becomes separable in objective and constraint functions. Second, we develop a general theory for a Lagrangian dual cutting plane (DCP) technique that is computationally-efficient when the number of constraints is sufficiently small as in this application. The DCP method enables efficient solution of the Lagrangian dual deleveraging problem. Third, when a nonzero duality gap exists, using the Lagrangian optimal solution, we develop the theory to determine successively-improving feasible portfolio solutions, computable via solving a sequence of convex optimization problems. These convex separable programs too can be globally-solved efficiently using the DCP technique. Consequently, our solution method allows efficient numerical evaluation of the complete trade-off frontier to develop managerial insights on setting policy parameters.

Applying the solution methodology, we test the optimal deleveraging model empirically using a portfolio constructed with ETF assets. While ETFs used here are more liquid assets, our results provide compelling evidence of the role leverage and margin policy parameters play under trading impact when down-sizing a portfolio with a large NAV in countering the effects of an impending market downturn. We show how the MV efficient frontiers are affected under deleveraging to stricter policies, and the serious over-estimation that occurs when liquidity impact is ignored. We test scenarios of increased liquidity costs and limited collateral availability to ascertain the value of prudent deleveraging policy, and provide insights for asset managers.

2. Model Development

Consider a one-period economy from date 0 to T with a given portfolio \( x_0 \in \mathbb{R}^n \) of \( n \) risky assets at date 0, where the initial asset (share) positions are denoted by \( x_{0j}, \ j = 1, \ldots, n \). At the initial
asset price vector $p_0 \in \mathbb{R}^n$, the leverage level and margin requirements of portfolio $x_0$ may violate the institution policy. Regardless, we assume a negative asset/market return regime during $(0, T]$ compels the portfolio $x_0$ to be deleveraged by reducing position sizes, with trade executions during the time period $(0, 1]$ to positions (decision) vector $x_1 \in \mathbb{R}^n$ at date 1, e.g., a monthly-portfolio with one-day deleveraging trade execution. While the decision $x_1$ is made at date 0, the execution of positions $x_0 \to x_1$ faces liquidity risks depending on the (continuous) trading trajectory $t \to x_{tj}$, $t \in (0, 1]$, corresponding to the trading rate $y_{tj}$ where $x_{1j} - x_{0j} = \int_0^1 y_{tj} dt$. If the trade-size $||x_1 - x_0||$ is large (as is the case with high NAV), portfolio trading may face significant permanent price impact due to market liquidity; if the position transition occurs at a high intensity, denoted by the instantaneous rate of trading $y_{tj} = \frac{dx_{tj}}{dt}$, then trading may also face temporary liquidity shortages, adversely affecting the trading price.

We assume $T \gg 1$, and thus, the market uncertainty in the execution period $(0, 1)$ is negligible relative to uncertainties of the period $[1, T]$. As such, the price of asset $j$ changes deterministically to $p_{tj}$ at time $t \in (0, 1]$, for $j = 1, \ldots, n$, due to the portfolio trading action. However, portfolio performance is evaluated at the terminal date $T$ captures uncertainties outside the deleveraging period based on price uncertainty using risk-averse utility. We shall assume that asset returns during $[1, T]$ to be normally-distributed with $r \sim \mathcal{N}(\mu, V)$, where $\mu = \mathbb{E}[r] \in \mathbb{R}^n$ is the mean vector and $V = \text{Var}[r] \in \mathbb{R}^{n \times n}$ is the covariance matrix.

As indicated, the price change of an asset during deleveraging has a permanent impact component that depends on the cumulative amount traded up until $t$; on the other hand, its temporary impact component depends on the rate at which the asset is traded and it is instantaneous and reversible. This single asset price evolutionary model in Carlin et al. (2007) was extended to a portfolio of multiple assets by Brown et al. (2010) - also see Chen et al. (2014):

$$
\begin{align*}
\frac{d}{dt} p_t &= p_0 + \Gamma(x_t - x_0) + \Lambda y_t, \quad t \in (0, 1) \\
p_1 &= p_0 + \Gamma(x_1 - x_0),
\end{align*}
$$

(1)

where the diagonal matrices $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n)$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, and $\gamma_j$ and $\lambda_j$ denote the (positive) permanent impact and temporary impact coefficients, respectively. The asset pricing model (1) assumes that during the short trading period there is no growth in the asset price, and the price changes are effected only by the investor’s trading action. Extensions with stochastic price variation during the trading period have been proposed.

Trading rate in an asset is positive, i.e., purchase, only if the asset has a short position at time 0; $y_{tj} \leq 0$ indicates asset sales for an initial long position, i.e., $x_{0j} > 0$. Stated formally:

\cite{Almgren2000, Almgren2003} split a single asset price into a market impact component due to trading and an unaffected price component that evolves according to the discrete arithmetic random walk. \cite{Gatheral2012} employed a geometric Brownian motion (GBM) for the unaffected asset prices.
**Assumption 1.** Given an asset $j = 1, \ldots, n$, $y_{tj} \geq 0$ for $t \in [0,1)$ and $x_{1j} \in [x_{0j}, 0]$ if $x_{0j} < 0$. Moreover, $y_{tj} \leq 0$ for $t \in [0,1)$ and $x_{1j} \in [0, x_{0j}]$ if $x_{0j} > 0$. If $x_{0j} = 0$, then $y_{tj} = 0$ for $t \in [0,1)$ and $x_{1j} = 0$.

A dynamic strategy is one in which the trade size depends on the stock price during execution of the order, such as in the case of a Delta-hedging strategy. For the special case of a single stock liquidation problem, Almgren and Chriss (2000) showed that under arithmetic Brownian motion, a static strategy is optimal - one that is determined in advance of trading, e.g. a constant trading rate, which is a volume-weighted average price (VWAP) approach. If the price process has no random term or the random component is independent of the current stock price, then a statically-optimal strategy will be dynamically-optimal for the problem of a single asset pure liquidation problem (Gatheral and Schied 2012). Motivated by this, since the price process during the execution period has no random term in our case, we follow Brown et al. (2010) and Chen et al. (2014) and employ the static constant-rate trading strategy, i.e., $y_j \equiv y_{tj} = x_{1j} - x_{0j}$. Then, the net cash 'generated' during the (short) deleveraging period is:

$$K(x_1) = -\int_0^1 p_i^\top y_i dt = -\int_0^1 [p_0^\top y + y^\top \Lambda y + ty^\top \Gamma y] dt$$

$$= -x_1^\top M x_1 + 2(Mx_0)^\top x_1 - x_0^\top M x_0 - p_0^\top (x_1 - x_0),$$

where $M := \Lambda + 0.5\Gamma$ is a positive definite matrix since $\Gamma$ and $\Lambda$ are positive definite, i.e., $\lambda_j, \gamma_j > 0$.

Denote the initial (cash) liability at day 0 of the portfolio by $L_0$. A positive or negative $L_0$ indicates an initial debt level or surplus cash position, respectively. Assuming a positive initial portfolio net wealth at date 0,

$$w_0 := p_0^\top x_0 - L_0 \ (> 0).$$

### 2.1. Leverage control

In the case of long-short portfolios, deleveraging may be ‘forced’ on the portfolio due to market conditions that threaten portfolio equity because such equity is used as collateral in creating short positions. To avoid forced liquidation (due to margin calls), additional capital may be borrowed exogenously, increasing portfolio liability. Portfolio managers resort to riskfree borrowing (hence, increasing debt to equity) to increase their exposure to risky assets in an attempt to increase target expected returns. In particular, modern Risk Parity strategies exploit this ‘financial leverage’ opportunity under exogenous borrowing to improve portfolio risk-adjusted performance. Using the portfolio’s debt to equity (D/E) ratio as a mechanism to guide portfolio performance is hereby referred to simply as leverage control.
On the other hand, degree of infusion of additional capital to increase collateral also depends on the specific assets in the portfolio since more-risky assets may contribute differently to collateral requirement than less-risky assets. Therefore, if the portfolio is of long-short type, then the short portfolio must have adequate collateral support from the long portfolio. The extent of short portfolio value relative to the long portfolio value must be managed for controlling margin requirements (or ‘portfolio leverage’). We shall consider both of these leverage control mechanisms as they provide different (but related) portfolio protections.

For example, for the initial holdings at date 0, D/E (financial) leverage ratio $\rho_0$ is:

$$\rho_0 := \frac{p_0^\top x_0 + L_0^+}{p_0^\top x_0 - L_0},$$  \hspace{1cm} (4)

where we use the following notation: $a^+ := \max\{a, 0\}$, $a^- = \max\{-a, 0\}$, and vector $b^-$ has its each component as $b^-_j$. Total portfolio debt comprising the cash liability and the short portfolio are considered in (4). Even if $\rho_0$ is a presently acceptable, portfolio deleveraging may still be required in anticipation of future market movement. In doing so, portfolio D/E leverage at date 1 must be brought under a prescribed level. For this, let:

$$L_1(x_1) = p_1^\top x_1^- + [L_0 - K(x_1)]^+$$

$$= p_1^\top x_1^- + [x_1^\top M x_1 + (p_0 - 2M x_0)^\top x_1 - \phi]^+, \hspace{1cm} (5)$$

where

$$\phi := p_0^\top x_0 - x_0^\top M x_0 - L_0. \hspace{1cm} (6)$$

The period-ending net equity (asset) position at date 1 is

$$A_1(x_1) = p_1^\top x_1 - [L_0 - K(x_1)]$$

$$= x_1^\top (\Gamma - M)x_1 + [(2M - \Gamma)x_0]^\top x_1 + \phi. \hspace{1cm} (7)$$

Thus, to satisfy the leverage ratio within a prescribed threshold $\rho$, the following constraint is imposed on portfolio rebalancing:

$$\frac{L_1(x_1)}{A_1(x_1)} \leq \rho. \hspace{1cm} (8)$$

For the initial portfolio $x_0$, denote the index set of long assets by $P$ and that of short assets by $N$. Then, the above constraint can be expressed as the following two constraints:

$$\rho A_1(x_1) + \sum_{j \in N} p_{1j} x_{1j} \geq 0$$

$$\rho A_1(x_1) + \sum_{j \in N} p_{1j} x_{1j} \geq x_1^\top M x_1 + (p_0 - 2M x_0)^\top x_1 - \phi, \hspace{1cm} (9)$$

where $p_1 = p_0 + \Gamma(x_1 - x_0)$.

---

3 Jacobs and Levy (2012) define portfolio leverage for long-short positions using the sum of absolute portfolio weights minus one, which is then incorporated into the objective utility function; also, see Jacobs and Levy (2013).
Remark: Noting (2), \( K(x_1) \) achieves a maximum (cash value), denoted by \( K_{\text{max}} \) when all long positions are liquidated, while all short positions remain unchanged, i.e., \( x_{1j} = 0, \forall j \in P \), and \( x_{1j} = x_{0j}, \forall j \in N \). Similarly, \( K(x_1) \) achieves a minimum cash position, denoted by \( K_{\text{min}} \), when all short positions are covered and all long positions remain unchanged, i.e., \( x_{1j} = 0, \forall j \in N \), and \( x_{1j} = x_{0j}, \forall j \in P \). That is,

\[
K_{\text{max}} = \sum_{j \in N} \left[ -M_{jj}(x_{0j})^2 + (2M_{jj}x_{0j} - p_{0j})x_{0j} \right] - x_0^\top Mx_0 + p_0^\top x_0,
\]

(10)

and

\[
K_{\text{min}} = \sum_{j \in P} \left[ -M_{jj}(x_{0j})^2 + (2M_{jj}x_{0j} - p_{0j})x_{0j} \right] - x_0^\top Mx_0 + p_0^\top x_0
\]

(11)

which yields \( \phi = K_{\text{max}} + K_{\text{min}} - L_0 \), see (6). Therefore, unless \( L_0 \in (K_{\text{min}}, K_{\text{max}}) \), one of the constraints in (9) can be removed. More specifically,

1. If \( K_{\text{max}} \leq L_0 \), then \( x_1^\top Mx_1 + (p_0 - 2Mx_0)^\top x_1 - \phi \geq L_0 - K_{\text{max}} \geq 0 \), and thus, the first constraint in (9) can be eliminated.

2. If \( K_{\text{min}} \geq L_0 \), then \( x_1^\top Mx_1 + (p_0 - 2Mx_0)^\top x_1 - \phi \leq L_0 - K_{\text{min}} \leq 0 \), and thus, the second constraint in (9) can be eliminated.

2.2. Margin control

Since short positions in a portfolio can lose more than its initial value (unlike long positions), short-sales must be covered by sufficient margin collateral to control ‘excessive shortsale risk’, see Edirisinghe (2007). That is, the total short position must be controlled not to exceed a certain fraction of the total (marginable) long positions.

Suppose an asset \( j \) contributes up to a maximum of \( \eta_j \) margin collateral per dollar invested in a long position in asset \( j \), where \( \eta_j \in [0, 1], \forall j \). Upon portfolio deleveraging, the rebalanced portfolio has a total marginable collateral of \( \sum_{j \in P} \eta_j p_{1j} x_{1j} \), and the short portfolio value is \( p_1^\top x_i = -\sum_{j \in N} p_{1j} x_{1j} \), given in absolute dollar terms. A portfolio margin control policy at level \( \zeta \) implies the constraint:

\[
-\sum_{j \in N} p_{1j} x_{1j} \geq \zeta.
\]

(12)

Since the denominator of (12) is nonnegative, for policy parameter \( \zeta > 0 \), short-sale restriction:

\[
\sum_{j \in N} p_{1j} x_{1j} + \zeta \sum_{j \in P} \eta_j p_{1j} x_{1j} \geq 0
\]

(13)

is a (nonconvex) quadratic constraint since \( p_1 = p_0 + \Gamma(x_1 - x_0) \).
2.3. Utility maximization

Subject to satisfying the leverage and margin constraints, the portfolio deleveraging is performed to maximize the expected utility of the net asset value at date \( T \), denoted by the random variable, \( A_T(x_1) \). We shall assume that the uncertainty evolves independent of the investor’s deleveraging action, over the period \([0,T]\), denoted by a price change of \( \Delta_j \) per share of asset \( j \). That is, \( \Delta_j = r_jp_{0j} \), where \( r_j \) is the asset’s random return over \([0,T]\). We approximate the random asset return over the period \([1,T]\) by \( r_j \) since \( T \gg 1 \). Accordingly, the random dollar return on the risky portfolio in \([1,T]\) is \( \sum_{j=1}^n \Delta_j x_{1j} \), while the risk-free component of the portfolio yields \( e^{r_0}[K(x_1) - L_0] \), where cash can be borrowed or lent at the continuously-compounded risk-free rate, \( r_0(>0) \), per period \( T \).

Defining the diagonal matrix \( P_0 = \text{diag}(p_{01}, \ldots, p_{0n}) \), thus,

\[
A_T(x_1) = A_1(x_1) + P_0 r^T x_1 + (e^{r_0} - 1)[K(x_1) - L_0].
\]  

Under normally-distributed \( r \) and CRRA utility function \( U \), maximizing \( \mathbb{E}[U(A_T(x_1))] \) is equivalent to determining an optimal trade-off between the portfolio variance:

\[
\mathbb{E}[A_T(x_1)] = x_1^T P_0 V P_0 x_1,
\]  

and the portfolio expected asset value at date \( T \):

\[
\mathbb{E}[A_T(x_1)] = A_1(x_1) + P_0 \mu^T x_1 + (e^{r_0} - 1)[K(x_1) - L_0] = x_1^T (\Gamma - M) x_1 + [P_0 \mu + (2M - \Gamma)x_0]^T x_1 + (e^{r_0} - 1)[K(x_1) - L_0] + \phi = x_1^T (\Gamma - e^{r_0} M) x_1 + [P_0 \mu + (2e^{r_0} M - \Gamma)x_0 - (e^{r_0} - 1)p_0]^T x_1 + e^{r_0} \phi.
\]  

Hence, the portfolio deleveraging model under CRRA risk aversion parameter \( \vartheta > 0 \), is:

\[
\max \quad \mathbb{E}[A_T(x_1)] - \vartheta x_1^T P_0 V P_0 x_1,
\]  

subject to the leverage and short-sale margin constraints in (9) and (14), respectively, under the policy-pair \((\rho, \zeta)\). We observe that this is a substantially-difficult optimization model to solve since the constraints have quadratic non-convexities and the objective may also be quadratic non-convex, coupled with the nonseparability introduced by the variance term. As such, we shall use an upper bounding separable risk metric on portfolio variance, see Edirisinghe (2007), a generalization of which is below.
PROPOSITION 1. Define the volatility-based risk function $\mathcal{R}_q(x_1)$, where $q$ is a given constant by:

$$\mathcal{R}_q(x_1) := \sum_{j=1}^{n} (p_{0j}\sigma_j|x_{1j}|)^q.$$ (18)

i. $\mathcal{R}_q(x_1)$ is positively homogeneous of degree $q$ in $x$. For $q \geq 1$, $\mathcal{R}_q(x_1)$ is convex in $x_1$.

ii. $\text{Var}[A_T(x_1)] \leq [\mathcal{R}_1(x_1)]^2$.

Proof. For some $\nu > 0$, $\mathcal{R}_q(\nu x_1) = \sum_{j=1}^{n} (P_j\sigma_j |\nu x_{1j}|)^q = \nu^q\mathcal{R}_q(x_1)$, thus proving the assertion (i).

The first partial derivative of $\mathcal{R}_q(x_1)$ w.r.t. $x_{1j}$ is $\nabla_j\mathcal{R}_q(x_1) = a_j p_{0j} \sigma_j |x_{1j}|^{q-1}$, where $a_j = +1$ if $x_{1j} \geq 0$, $a_j = -1$ if $x_{1j} < 0$. Then, the Hessian $\nabla^2 \mathcal{R}_q(x_1)$ is a diagonal matrix where the $j^{th}$ diagonal element is $(a_j)^2q(q-1)P_j\sigma_j |x_{1j}|^{q-2}$, which is nonnegative if $q \geq 1$. Thus, $\nabla^2 \mathcal{R}_q(x_1)$ is positive semidefinite, implying that $\mathcal{R}_q(x_1)$ is convex for $q \geq 1$. To show part (ii), define the random variable $\xi_j := p_{0j}x_{1j}$. Then, $\text{Var}[A_T(x_1)] = \text{Var}\left(\sum_{j=1}^{n} \xi_j \right)$. Denoting the standard deviation of $\xi_j$ by $\hat{\sigma}_j$ and the correlation between $\xi_i$ and $\xi_j$ by $\hat{\rho}_{ij}$,

$$\text{Var}\left(\sum_{j=1}^{n} \xi_j \right) = \sum_{j=1}^{n} \hat{\sigma}_j^2 + 2 \sum_{(i,j), i \neq j} \hat{\sigma}_i \hat{\sigma}_j \hat{\rho}_{ij} \leq \sum_{j=1}^{n} \hat{\sigma}_j^2 + 2 \sum_{(i,j), i \neq j} \hat{\sigma}_i \hat{\sigma}_j = \left(\sum_{j=1}^{n} \hat{\sigma}_j^2\right)^2.$$ (19)

Since $(\hat{\sigma}_j)^2 = \text{Var}(\xi_j) = [p_{0j}x_{1j}\sigma_j]^2$, and the prices are nonnegative, we have $\hat{\sigma}_j = p_{0j}\sigma_j |x_{1j}|$. □

Therefore, $\mathcal{R}_1(x_1)$ is an upper bounding function (on standard deviation) that is positively homogeneous, convex, and separable in portfolio positions. Geometrically, $\mathcal{R}_1(x_1)$ is a polyhedral convex cone with apex at the origin. The expected utility maximization in (17) can be replaced with the more-conservative, separable (hence computationally-friendly), risk function $\mathcal{R}_1(x_1)$:

$$\max_{x_1} \mathbb{E}[A_T(x_1)] - \theta \left[ \sum_{j \in P} p_{0j}\sigma_j x_{1j} - \sum_{j \in N} p_{0j}\sigma_j x_{1j} \right]^2 \leq \mathbb{E}[A_T(x_1)] - \theta x_1^\top \mathbf{P}_0 \mathbf{V} \mathbf{P}_0 x_1.$$ (20)

Equivalently, for a specified maximum portfolio standard deviation risk of $s > 0$, and given leverage and margin policy parameter-pair $(\rho, \zeta)$, the resulting (correlation-free) separable deleveraging optimization (SDO) model is:

$$F(\rho, \zeta, s) := \max_{x_1} x_1^\top (\Gamma - \epsilon \epsilon^\top M) x_1 + [\mathbf{P}_0 \mu + (2\epsilon \epsilon^\top M - \Gamma) x_0 - (\epsilon \epsilon^\top - 1) p_{0j}]^\top x_1$$

s.t. $\sum_{j \in P} p_{0j}\sigma_j x_{1j} - \sum_{j \in N} p_{0j}\sigma_j x_{1j} \leq s$

$\rho A_1(x_1) + \sum_{j \in N} p_{1j} x_{1j} \geq 0$

$\rho A_1(x_1) + \sum_{j \in N} p_{1j} x_{1j} - x_1^\top M x_1 - (p_0 - 2M x_0)^\top x_1 + \phi \geq 0$

$\sum_{j \in N} p_{1j} x_{1j} + \zeta \sum_{j \in P} \eta_j p_{1j} x_{1j} \geq 0$

$0 \leq x_{1j} \leq x_{0j}, \ j \in P; \ x_{0j} \leq x_{1j} \leq 0, \ j \in N.$ (21)
Proposition 2. For the initial portfolio \((x_0, L_0)\), assume that \(\phi \geq 0\), where \(\phi\) is defined in (6). Then, the feasible set of SDO model (21) is nonempty. In particular, the fully-liquidated portfolio \(x_1 = 0\) is feasible in (21).

For the current portfolio Let an optimal solution of (21) be denoted by \(x_1^*\). We are particularly interested in the trade-off frontier between \(E[A_T(x_1^*)] \equiv [F(\rho, \zeta, s) + e^r_0\phi]\) and portfolio standard deviation \(\sigma_P(x_1^*) := \sqrt{x_1^{*\top}P_0VP_0x_1^*} \leq s\), for fixed policy \((\rho, \zeta)\) as \(s\) is varied, as well as the effect of changing the latter policy on the frontier.

Since \(p_1 = p_0 + \Gamma(x_1 - x_0)\), observe that the SDO model (21) has three quadratic separable constraints and one linear constraint, in addition to the lower and upper limits on variables. Note that the margin constraint is nonconvex, and also depending on the liquidity impact coefficient-values, leverage constraint and the objective function can be indefinite (or nonconvex). Solution of such optimization models is computationally tedious, particularly when \(n\) is large. The linear-time algorithm developed for the singly-constrained version in Edirisinghe and Jeong (2017) may be extended based on the constraint surrogation method developed in Edirisinghe and Jeong (2018); but, such an approach may not prove to be efficient.

Alternatively, our approach is based on first solving a certain Lagrangian dual by developing an efficient dual cutting plane algorithm to compute an upper bound on \(F(\rho, \zeta, s)\). The necessary theory is developed in Section 3. Subsequently, we derive an algorithm to generate lower bounding feasible portfolios for (21), so that a near-optimal feasible deleveraging strategy can be obtained.

3. Theory of Dual Cutting-Planes (DCP)

To develop our dual cutting-plane technique, consider (21) in a generalized setting as follows, i.e., the primal optimization problem:

\[
\max_{x \in X} \{f(x) : g_j(x) \leq 0, \ j = 1, \ldots, m\}, \tag{22}
\]

where \(X \subset \mathbb{R}^n\). We shall assume that (22) is feasible. Define the Lagrangian of (22) using the multipliers \(\pi_j, \ j = 1, \ldots, m\), of the inequality constraints:

\[
L(x, \pi) := f(x) - \sum_{j=1}^m \pi_j g_j(x). \tag{23}
\]

The Lagrangian dual problem of (22) is:

\[
L^{**} := \min_{\pi \in \Pi} L^*(\pi) \tag{24}
\]

where

\[
L^*(\pi) := \max_{x \in X} L(x, \pi) \quad \text{and} \quad \Pi = \{\pi \in \mathbb{R}^m : \pi_j \geq 0, \ j = 1, \ldots, m\}. \tag{25}
\]
Feasibility of (22) ensures that (24) is bounded, i.e., at an optimal solution \( \pi^* \), \( L^{**} = L^*(\pi^*) > -\infty \), and thus, \( \pi_j^* < \infty, \forall j \). In determining the dual optimum value \( L^{**} \), the outer dual iteration of the min-max problem (24) requires a computationally-intensive search over the unbounded region \( \Pi \) in (25), for instance, using subgradient optimization. Instead, in this paper, we propose an efficient scheme based on a sequence of dual cutting planes on a compact region that progressively shrinks the dual space to determine the dual optimum efficiently. To the best of our knowledge, such an approach has not been taken in the literature.

3.1. Mapping multipliers

Consider the transformation (mapping) \( T = (T_1, \ldots, T_m) \) such that \( T_j : \mathbb{R}^m \to \mathbb{R} \), where:

\[
\theta_j = T_j(\pi) := \frac{\pi_j}{1 + \sum_{k=1}^m \pi_k}, \quad j = 1, \ldots, m. \tag{26}
\]

**Proposition 3.** Suppose \( \pi \in \Pi \). Then, the following properties hold for \( T \):

(P1) \( \theta \in \Theta \) where the open set:

\[
\Theta := \{ \theta \in \mathbb{R}^m : \sum_{j=1}^m \theta_j < 1, \theta \geq 0 \}. \tag{27}
\]

(P2) \( T \) has the unique inverse mapping, \( T^{-1} : \Theta \to \Pi \), given by:

\[
\pi_j = T_j^{-1}(\theta) := \frac{\theta_j}{1 - \sum_{k=1}^m \theta_k}, \quad j = 1, \ldots, m. \tag{28}
\]

On the open boundary of \( \Theta \), \( \pi_j \to \infty, \forall j \); otherwise, \( \pi \) is finite.

(P3) \( T \) and \( T^{-1} \) are one-to-one and onto mappings with \( T(\Pi) = \Theta \) and \( T^{-1}(\Theta) = \Pi \).

**Proof.** Since \( \pi_j \geq 0 \), (26) gives \( \theta_j \geq 0 \). Next, \( \sum_j \theta_j = \sum_j \pi_j / (1 + \sum_j \pi_j) < 1 \) since \( \sum_j \pi_j \geq 0 \), i.e., \( \theta \in \Theta \).

Next, rewriting (26), for fixed \( \theta \in \Theta \):

\[
(1 - \theta_j)\pi_j - \sum_{k \neq j, k=1}^m \theta_j \pi_k = \theta_j, \quad j = 1, \ldots, m. \tag{29}
\]

To show that (29) admits a unique solution in \( \Pi \), observe that (29) is the system of equalities:

\[
(I - \theta 1^T)\pi = \theta, \quad \tag{30}
\]

where \( I - \theta 1^T \) is non-singular for any \( \theta \in \Theta \). Applying Sherman-Morrison (1949) formula:

\[
(I - \theta 1^T)^{-1} = I + \frac{\theta 1^T}{1 - 1^T \theta} \tag{31}
\]

which exists since \( 1^T \theta < 1 \) due to \( \theta \in \Theta \). Therefore, the unique solution of (26) for fixed \( \theta \in \Theta \) is:

\[
\pi = (I - \theta 1^T)^{-1}\theta = \theta + \left( \frac{\theta 1^T}{1 - 1^T \theta} \right) \theta = \frac{\theta}{1 - 1^T \theta} \in \Pi. \tag{32}
\]

Therefore, we have \( T^{-1}(T(\pi)) = \pi \) and \( T(T^{-1}(\theta)) = \theta \). Hence, there is one-to-one correspondence between \( \Pi \) and \( \Theta \). \( \square \)
3.2. Dual reformulation

Define the function:
\[
L(x,\theta) := (1 - \sum_{j=1}^{m} \theta_j) f(x) - \sum_{j=1}^{m} \theta_j g_j(x),
\]  
(33)

and consider the problem:
\[
L^{**} := \min_{\theta \in \Theta} L^*(\theta)
\]  
(34)

where
\[
L^*(\theta) := \max_{x \in X} L(x,\theta).
\]  
(35)

We shall derive a crucial relationship between \(L^{**}\) in (24) and \(L^{**}\) in (34). Let an optimal solution of the Lagrangian dual (24) be \((\bar{x}, \bar{\pi})\). Define: \(\bar{\theta} = T(\bar{\pi})\). Since \(\bar{\pi} \geq 0\), from (P1), we have \(\bar{\theta} \geq 0\) and \(\sum_{j=1}^{m} \bar{\theta}_j < 1\), i.e., \(\bar{\theta} \in \Theta\). Moreover, since \(\bar{\pi} = T^{-1}(\bar{\theta})\),
\[
L^{**} = L(\bar{x}, T^{-1}(\bar{\theta})) = f(\bar{x}) - \sum_{j=1}^{m} T^{-1}_j(\bar{\theta}) \bar{\theta}_j g_j(\bar{x})
\]
\[
= f(\bar{x}) - \sum_{j=1}^{m} \frac{\bar{\theta}_j}{1 - \sum_{k=1}^{m} \bar{\theta}_k} g_j(\bar{x})
\]
\[
= \frac{1}{1 - \sum_{j=1}^{m} \bar{\theta}_j} L(\bar{x}, \bar{\theta}).
\]  
(36)

PROPOSITION 4.
\[
L^{**} = \min_{\theta \in \Theta} h(\theta) \quad \text{where} \quad h(\theta) := \frac{L^*(\theta)}{1 - \sum_{j=1}^{m} \theta_j}.
\]  
(37)

Proof. Suppose \(x^*(\theta)\) solves (35) for fixed \(\theta \in \Theta\). Then,
\[
L^{**} \leq L(x^*(\theta), T^{-1}(\bar{\theta})) = f(x^*(\theta)) - \sum_{j=1}^{m} T^{-1}_j(\bar{\theta}) \bar{\theta}_j g_j(x^*(\theta))
\]
\[
= f(x^*(\theta)) - \sum_{j=1}^{m} \frac{\theta_j}{1 - \sum_{k=1}^{m} \theta_k} g_j(x^*(\theta))
\]
\[
= \frac{L(x^*(\theta), \theta)}{1 - \sum_{j=1}^{m} \theta_j} = \frac{L^*(\theta)}{1 - \sum_{j=1}^{m} \theta_j},
\]  
(38)

and thus, \(h(\theta) \geq L^{**}\) for any \(\theta \in \Theta\). Next, it is only necessary to observe that there exists \(\bar{\theta} \in \Theta\) and \(\bar{x}(\bar{\theta}) \in X\) such that \(h(\bar{\theta}) = L^{**}\) is held due to (36). \(\square\)

Therefore, instead of solving the original dual problem in (24), we may solve the equivalent formulation in (37), in which \(h(\theta)\) is defined over a compact (open) set \(\Theta\). We develop an iterative cutting plane technique on \(\Theta\) so that (37) can be computed efficiently.
3.3. Dual cutting-plane (DCP) method

Suppose at some iteration \(t\), given \(\theta^t \in \Theta\), an optimal solution of (35) is \(x^t \equiv x^*(\theta^t)\), i.e.,

\[
\mathcal{L}^*(\theta^t) = \mathcal{L}(x^t, \theta^t) = (1 - \sum_{j=1}^{m} \theta_j^t) f(x^t) - \sum_{j=1}^{m} \theta_j^t g_j(x^t).
\]

The next iterate \(\theta^{t+1} \in \Theta\) must be chosen such that \(h(\theta^{t+1}) < h(\theta^t)\) toward solving the minimization in (37); however, since \(x^t \in X\) is feasible in (35) for \(\theta^{t+1}\), we must have:

\[
\mathcal{L}^*(\theta^{t+1}) \geq \mathcal{L}(x^t, \theta^{t+1})
\]

which yields

\[
h(\theta^{t+1}) = \frac{\mathcal{L}^*(\theta^{t+1})}{1 - \sum_{j=1}^{m} \theta_j^{t+1}} \\
\geq \frac{\mathcal{L}(x^t, \theta^{t+1})}{1 - \sum_{j=1}^{m} \theta_j^{t+1}} = f(x^t) - \sum_{j=1}^{m} \frac{\theta_j^{t+1}}{1 - \sum_{k=1}^{m} \theta_k^{t+1}} g_j(x^t).
\]

Proposition 5. A new iterate \(\theta\) cannot lead to an improvement in the objective value over \(h(\theta^t)\) unless the following linear inequality holds:

\[
\sum_{j=1}^{m} \alpha_j^t \theta_j > \beta^t,
\]

where the coefficients \(\alpha^t \in \mathbb{R}^m\) and \(\beta^t \in \mathbb{R}\) are given by:

\[
\alpha_j^t := g_j(x^t) + \beta^t, \quad j = 1, \ldots, m \quad \text{and} \quad \beta^t := f(x^t) - h(\theta^t) = \frac{\sum_{j=1}^{m} \theta_j^t g_j(x^t)}{1 - \sum_{j=1}^{m} \theta_j^t}.
\]

Proof. Combining \(h(\theta^{t+1}) < h(\theta^t)\) with (40), for \(\theta \equiv \theta^{t+1}:

\[
f(x^t) - \sum_{j=1}^{m} \frac{\theta_j}{1 - \sum_{k=1}^{m} \theta_k} g_j(x^t) < h(\theta^t),
\]

which yields:

\[
\sum_{j=1}^{m} \theta_j g_j(x^t) > [f(x^t) - h(\theta^t)] \left[1 - \sum_{j=1}^{m} \theta_j \right] \\
or, \sum_{j=1}^{m} [g_j(x^t) + f(x^t) - h(\theta^t)] \theta_j > f(x^t) - h(\theta^t).
\]

Noting (42), the proof is completed. □

Observe that for \(\theta = \theta^t\), we have \(\sum_{j=1}^{m} \alpha_j^t \theta_j^t = \beta^t\). Thus, the dual cutting plane generated on \(\Theta\) at iteration \(t\) is: \(\alpha^T \theta \leq \beta^t\). Hence, the updated (shrunk) dual feasible region is:

\[
\Theta_t := \Theta \setminus \bigcup_{\tau=1}^{t-1} \{ \theta \in \mathbb{R}^m : \alpha^T \theta \leq \beta^\tau \}.
\]
While $\Theta$ is an $m$-dimensional simplex (with apex at the origin), observe that $\Theta_t$ can be a general polytope in $m$-dimensions. The next iterate $\theta^{t+1}$ is set to be the centroid of $\Theta_t$, denoted by $\theta^{t+1} = C(\Theta_t)$ for $t = 0, 1, 2, \ldots$, where we have set $\Theta_0 \equiv \Theta$; see the two-dimensional illustration in Figure 1. The termination criterion for improving the (dual) solution of (37) is when $|\theta^{t+1} - \theta^t| < \varepsilon$ for a specified (scaled) tolerance $\varepsilon > 0$. To determine the centroid of $\Theta_t$, we employ a standard technique, e.g., Cut-Off Polyhedron (COP) method (Nakagawa et al. 1984), see Appendix A.

4. Theory of Feasible Portfolios

The application of the preceding DCP method on the deleveraging problem (21) yields an upper bound $F_U(\rho, \zeta, s) \geq F(\rho, \zeta, s)$, associated with an optimal primal-dual pair, $(x^U_1, \pi^U)$. If $x^U_1$ is feasible in (21), it indeed is a global optimal portfolio solution; otherwise, there may exist a nonzero duality gap. Then, we generate an improving-sequence of lower bounds, starting with the solution estimate $x^U_1$ such that a tight lower bound $F_L(\rho, \zeta, s) \leq F(\rho, \zeta, s)$ is available with a feasible portfolio solution $x^L_1$.

Toward this, consider SDO model in (21) in the following stylized format of a separable quadratic program with an indefinite objective function and $m$ indefinite quadratic constraints and variable bounds:

$$Z^* = \max_{l \leq x \leq u} f(x) \equiv \sum_{j \in J^+_0} d_{0j}(x_j)^2 + \sum_{j \in J^-_0} d_{0j}(x_j)^2 + c^T_0 x$$

$$\text{s.t. } \sum_{j \in J^+_i} d_{ij}(x_j)^2 + \sum_{j \in J^-_i} d_{ij}(x_j)^2 + c_i^T x \leq b_i, \ i = 1, \ldots, m, \quad (44)$$

where $d_{0j} > 0$ for $j \in J^+_0$ and $J^+_0 := \{1, \ldots, n\} \setminus J^-_0$; moreover, $d_{ij} < 0$ for $j \in J^-_i$ and $J^+_i := \{1, \ldots, n\} \setminus J^-_i$, $i = 1, \ldots, m$. 

---

**Figure 1**  
Lagrangian dual cutting plane approach to optimization
At the $k^{th}$ iteration of generating a lower approximation, let $x_k = y_k$ be a given portfolio solution. at the initial approximation ($k = 0$), we set $y^0 = x_1^\top$. Using Taylor’s first-order approximation (of indefinite terms) of (44) at $y^k$:

\[
\sum_{j \in J_0^+} d_{0j}(x_j)^2 \geq \sum_{j \in J_0^+} [d_{0j}(y_j^k)^2 + 2d_{0j}y_j^k(x_j - y_j^k)] = \sum_{j \in J_0^+} 2(d_{0j}y_j^k)x_j - \sum_{j \in J_0^+} d_{0j}(y_j^k)^2 \right \}
\]

and for $i = 1, \ldots, m$:

\[
\sum_{j \in J_i^-} d_{ij}(x_j)^2 \leq \sum_{j \in J_i^-} [d_{ij}(y_j^k)^2 + 2d_{ij}y_j^k(x_j - y_j^k)] = \sum_{j \in J_i^-} 2(d_{ij}y_j^k)x_j - \sum_{j \in J_i^-} d_{ij}(y_j^k)^2. \tag{45}
\]

Then, consider the following problem:

\[
f_L(y^k) := \max_{t \leq x \leq u} \sum_{j \in J_0^+} d_{0j}(x_j)^2 + c_0^\top x + 2 \sum_{j \in J_0^+} (d_{0j}y_j^k)x_j \\
\text{s.t.} \quad \sum_{j \in J_i^-} d_{ij}(x_j)^2 + c_i^\top x + 2 \sum_{j \in J_i^-} (d_{ij}y_j^k)x_j \leq b_i + \sum_{j \in J_i^-} d_{ij}(y_j^k)^2, \quad i = 1, \ldots, m. \tag{46}
\]

Then, it is straightforward to claim that $Z^* \geq f_L(y^k) - \sum_{j \in J_0^+} d_{0j}(y_j^k)^2$ since the inequalities (45) ensure that the feasible region of (46) is a ‘restriction’ over that of (44) and the objective of the latter is a lower approximation on the former. Moreover, (46) is a separable convex program, which can be solved to global optimality using the DCP method presented earlier since strong Lagrangian duality holds for (46). Let a (global) optimum solution of (46) be denoted by $y^{k+1}$. The fact that a monotonic sequence of lower bounds can be generated is claimed in the following result.

**Proposition 6.** Suppose the fixed $y^k$ is chosen feasible in (44), and let an optimal solution of (46) be denoted by $y^{k+1}$. Then, $y^{k+1}$ is feasible in (44), and $Z^* \geq f(y^{k+1}) \geq f(y^k)$. Moreover, $f(y^{k+1}) - f(y^k) \geq \sum_{j \in J_0^+} d_{0j}(y_j^{k+1} - y_j^k)^2 \geq 0$.

**Proof.** First, given $y^k$ is feasible in (44), and since the first-order approximation is performed at $y^k$ to obtain the constraints of (46), $y^k$ must also be feasible in (46). Then, due to optimality of $y^{k+1}$ in (46),

\[
f_L(y^k) = \sum_{j \in J_0^+} d_{0j}(y_j^{k+1})^2 + c_0^\top y^{k+1} + 2 \sum_{j \in J_0^+} (d_{0j}y_j^k)y_j^{k+1} \\
\geq \sum_{j \in J_0^+} d_{0j}(y_j^k)^2 + c_0^\top y^k + 2 \sum_{j \in J_0^+} (d_{0j}y_j^k)y_j^k \\
= f(y^k) + \sum_{j \in J_0^+} d_{0j}(y_j^k)^2. \tag{47}
\]
Next,

\[ f(y^{k+1}) = \sum_{j \in J^+} d_{0j} (y_j^{k+1})^2 + \sum_{j \in J^-} d_{0j} (y_j^{k+1})^2 + c_0^T y^{k+1} \]

\[ = f_L(y^k) - 2 \sum_{j \in J^+} (d_{0j} y_j^k) y_j^{k+1} + \sum_{j \in J^+} d_{0j} (y_j^{k+1})^2 \]

\[ \geq f(y^k) + \sum_{j \in J^+} d_{0j} (y_j^k)^2 - 2 \sum_{j \in J^+} (d_{0j} y_j^k) y_j^{k+1} + \sum_{j \in J^+} d_{0j} (y_j^{k+1})^2 \]

\[ = f(y^k) + \sum_{j \in J^+} d_{0j} (y_j^{k+1} - y_j^k)^2 \]

\[ \geq f(y^k), \]

where the first inequality follows due to (47), and the second inequality holds due to \((y_j^{k+1} - y_j^k)^2 \geq 0\) and \(d_{0j} > 0, j \in J^+_0\). This completes the proof. □

Therefore, at some iteration \(k\) when \(y^k\) is feasible in (44), the convex model (46) not only generates a solution \(y^{k+1}\) feasible in (44), but also it provides an improved lower bound, i.e.,

\[ f(y^{k+1}) > f(y^k), \]

provided at least one component of \(y_j^{k+1}\) is different from \(y_j^k\) for some \(j \in J^+_0\). That is, so long as the solution of (46) varies iteratively, a strictly-improving sequence of lower bounds on (46) is generated. The sequence must converge since the optimal value \(Z^*\) of (44) provides a finite upper bound on the monotonic sequence \(\{f(y^k)\}\); moreover,

**Proposition 7.** Suppose an optimal solution of (44) is denoted by \(x^*\), i.e., \(Z^* \equiv f(x^*)\). Let an optimal solution of (46) with \(y^k = x^*\) be given by \(x^{**}\). Then, \(f(x^{**}) = f^*\), i.e., \(x^{**}\) solves (44).

**Proof.** Due to Proposition 6, we must have \(f(x^{**}) \geq f(x^*)\). However, since \(x^{**}\) is feasible in (44), \(f(x^{**}) \leq f^*\) holds, and thus, it follows that \(f(x^{**}) = f^*\). □

**Remark:** For \(k = 0\), \(y^0 = x^u\) will be infeasible in (44), for if not, \(x^u\) is an optimal deleveraged portfolio in (21). Suppose the resulting model (46) for the lower bound \(f_L(y^0)\) is also infeasible. Since the fully-liquidated portfolio \(x_1 = 0\) is feasible in (44), see Proposition 2, in this case, we set \(y^0 = 0\), so that (46) is guaranteed to be feasible.

On the other hand, suppose the model (46) resulting from \(y^0 = x^u\) is feasible, but its optimal solution \(y^1\) is infeasible in (44). In this case, we reset \(y^0 = 0\) so that the optimal solution \(y^1\) is guaranteed to be feasible in (44).
4.1. Computing bounds

To apply the DCP method to compute dual upper bounds, all of the constraints of (21) Lagrangian relaxed except for the bounds on variables. Using the stylized problem in (44), for some $\theta^t \in \Theta$ at iteration $t$,

$$
L^*(\theta^t) = \max_{l \leq x \leq u} \left( 1 - \sum_{i=1}^{m} \theta^t_i \right) \left[ \sum_{j \in J^+_0} d_{0j} (x_j)^2 + \sum_{j \in J^-_0} d_{0j} (x_j)^2 + c_0^T x \right] - \sum_{i=1}^{m} \theta^t_i \left[ \sum_{j \in J^+_i} d_{ij} (x_j)^2 + \sum_{j \in J^-_i} d_{ij} (x_j)^2 + c_i^T x - b_i \right].
$$

(48)

Its solution $x^t_1$ is obtained directly by solving a sequence of univariate quadratic problems. Then, the cut $\sum_{i=1}^{m} \alpha^t_i \theta_i \leq \beta^t$ is imposed on $\Theta$ to obtain the reduced search space $\Theta_t$, see (42)-(43). Setting $\theta^{t+1} = C(\Theta_t)$, the centroid of $\Theta_t$, (48) is re-solved, and the procedure is terminated when $||\theta^{t+1} - \theta^t|| < \varepsilon$, with solution $x^t_1$ and upper bound $F_U(\rho, \zeta, s)$ in hand.

To compute the lower bound, the model (46) is Lagrangian relaxed in the same way as (48), but it is now a pure convex quadratic program. Its Lagrangian dual optimum is determined using the DCP method as in the preceding paragraph. However, in this case, the DCP-based solution method determines the optimal value $f_L(y^k)$, and its optimal solution $y^{k+1}$, because the model (46) has no Lagrangian duality gap. Then, $f(y^k)$ is an improved lower bound on the deleveraging model, SDO. The lower bound is further improved by forming a new lower bounding model using $y^{k+1}$, which is solved using yet another iterative scheme of the DCP method. The process is terminated when $f(y^{k+1}) - f(y^k) < \varepsilon$, say at some iteration $K$. Therefore, the (best) feasible deleveraged portfolio $y^K$ is associated with an optimum-quality guarantee, w.r.t. $F(\rho, \zeta, s)$ in (21), as measured by the relative error:

$$
\mathcal{E} := \frac{F_U(\rho, \zeta, s) - f(y^K)}{1 + |f(y^K)|}.
$$

(49)

Note that in the process of determining $\mathcal{E}$, the iterative Lagrangian-based DCP solution method has been used $K + 1$ times. In the application reported next, we compute the deleveraged portfolio $y^K$ with the optimality guarantee $1 - \mathcal{E}$.

5. Portfolio Deleveraging Application

We shall use the data set on nine sector-ETFs (ticker symbols XLB, XLE, XLF, XLI, XLK, XLP, XLU, XLV, and XLY), as reported in Edirisinghe et al. (2018), with trading period set to one day, and the trade holding (rebalancing period) set to $T = 21$ days (one month). These ETFs cover the full-breadth of the S&P 500 market index. First, the market impact parameters are estimated using the millisecond TAQ data for these 9 ETFs using the time period, Jan 01-31, 2015. The
estimation methodology we employed is described in Edirisinghe et al. (2018), a summary of which is in Appendix EC.2. The estimated parameters $\gamma$ and $\lambda$ are in Table 1. In the same table, we report the forecasted mean and covariance parameters of asset and the market index returns, for the 21-day period Aug 03-31, 2015, including those for the market index proxy ETF ticker, SPY.

Initial portfolio (net risky) investment is $p_0^\top x_0 = $2m, and the initial liability is $L_0 = $1m, hence the initial wealth is $w_0 = p_0^\top x_0 - L_0 = $1m. We set the deleveraging of the initial portfolio $x_0$ as Aug 03, 2015, also reported in Table 1 as portfolio dollar weights, computed by $p_0^j x_0^j / p_0^\top x_0$, $\forall j$. Thus, the long portfolio is $p_0^\top x_0^+ = $3m, while the short portfolio value is $p_0^\top x_0^- = $1m. This initial portfolio allocations reflect the market outlook predicted in the previous month, where the actual returns for the period Jul 01-31, 2015, are also reported in Table 1. Initial positions are largely long consistent with the current conditions, including the positive market index return (+1.5%) for the period Jul 01-31. However, a negative market performance (−0.3%) is predicted for Aug 03-31. Therefore, portfolio $x_0$ is going to be deleveraged towards a new (optimal) portfolio $x_1$ during the trading day of Aug 03, and $x_1$ is held unchanged through Aug 31. Annualized risk-free rate is $r_0 = 2\%$ (used to compute the monthly-compounded rate).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>XLB</th>
<th>XLE</th>
<th>XLF</th>
<th>XLI</th>
<th>XLK</th>
<th>XLP</th>
<th>XLU</th>
<th>XLV</th>
<th>XLY</th>
<th>SPY</th>
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<tr>
<td>$\gamma \times 10^{-6}$</td>
<td>0.6361</td>
<td>0.7157</td>
<td>0.0127</td>
<td>0.2284</td>
<td>0.0578</td>
<td>0.0294</td>
<td>0.1797</td>
<td>0.3204</td>
<td>0.5599</td>
<td>-</td>
</tr>
<tr>
<td>$\lambda \times 10^{-6}$</td>
<td>4.3737</td>
<td>6.5823</td>
<td>0.3776</td>
<td>2.8244</td>
<td>1.2855</td>
<td>3.2523</td>
<td>4.2074</td>
<td>4.8528</td>
<td>4.6215</td>
<td>-</td>
</tr>
<tr>
<td>Mean ($\mu$)</td>
<td>-0.038</td>
<td>-0.046</td>
<td>0.009</td>
<td>-0.019</td>
<td>-0.009</td>
<td>0.010</td>
<td>-0.007</td>
<td>0.013</td>
<td>0.015</td>
<td>-0.003</td>
</tr>
<tr>
<td>StDev ($\sigma$)</td>
<td>0.023</td>
<td>0.018</td>
<td>0.013</td>
<td>0.012</td>
<td>0.020</td>
<td>0.027</td>
<td>0.032</td>
<td>0.013</td>
<td>0.013</td>
<td>0.012</td>
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Asset correlations ($\rho$)

<table>
<thead>
<tr>
<th>Asset correlations ($\rho$)</th>
<th>XLB (Basic Materials)</th>
<th>XLE (Energy)</th>
<th>XLF (Financials)</th>
<th>XLI (Industrial Goods)</th>
<th>XLK (Technology)</th>
<th>XLP (Consumer Staples)</th>
<th>XLU (Utilities)</th>
<th>XLV (Health Care)</th>
<th>XLY (Consumer Discre.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>XLB</td>
<td>1</td>
<td>0.874</td>
<td>0.105</td>
<td>0.512</td>
<td>-0.120</td>
<td>-0.569</td>
<td>0.461</td>
<td>-0.091</td>
<td>0.080</td>
</tr>
<tr>
<td>XLE</td>
<td>0.874</td>
<td>1</td>
<td>0.040</td>
<td>0.509</td>
<td>-0.169</td>
<td>-0.363</td>
<td>-0.506</td>
<td>0.459</td>
<td>0.116</td>
</tr>
<tr>
<td>XLF</td>
<td>0.105</td>
<td>0.040</td>
<td>1</td>
<td>0.699</td>
<td>0.839</td>
<td>0.290</td>
<td>0.207</td>
<td>0.625</td>
<td>0.837</td>
</tr>
<tr>
<td>XLI</td>
<td>0.512</td>
<td>0.509</td>
<td>0.839</td>
<td>1</td>
<td>0.660</td>
<td>0.196</td>
<td>0.143</td>
<td>0.848</td>
<td>0.839</td>
</tr>
<tr>
<td>XLK</td>
<td>-0.120</td>
<td>-0.169</td>
<td>0.290</td>
<td>0.660</td>
<td>1</td>
<td>0.473</td>
<td>0.498</td>
<td>0.586</td>
<td>0.881</td>
</tr>
<tr>
<td>XLP</td>
<td>-0.569</td>
<td>-0.363</td>
<td>0.839</td>
<td>0.196</td>
<td>0.473</td>
<td>1</td>
<td>0.883</td>
<td>0.303</td>
<td>0.605</td>
</tr>
<tr>
<td>XLU</td>
<td>0.461</td>
<td>-0.506</td>
<td>0.290</td>
<td>0.143</td>
<td>0.498</td>
<td>0.883</td>
<td>1</td>
<td>0.197</td>
<td>0.526</td>
</tr>
<tr>
<td>XLV</td>
<td>-0.091</td>
<td>-0.363</td>
<td>0.625</td>
<td>0.848</td>
<td>0.586</td>
<td>0.303</td>
<td>0.197</td>
<td>1</td>
<td>0.819</td>
</tr>
<tr>
<td>XLY</td>
<td>0.080</td>
<td>0.116</td>
<td>0.837</td>
<td>0.839</td>
<td>0.881</td>
<td>0.605</td>
<td>0.526</td>
<td>0.819</td>
<td>0.884</td>
</tr>
</tbody>
</table>

Initial portfolio weights

<table>
<thead>
<tr>
<th>Initial portfolio weights</th>
<th>-12.5%</th>
<th>-12.5%</th>
<th>-12.5%</th>
<th>-12.5%</th>
<th>25.0%</th>
<th>37.5%</th>
<th>37.5%</th>
<th>12.5%</th>
<th>37.5%</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open price $p_0$ (Aug 03)</td>
<td>44.45</td>
<td>66.57</td>
<td>19.75</td>
<td>52.43</td>
<td>41.41</td>
<td>48.41</td>
<td>41.70</td>
<td>74.80</td>
<td>78.22</td>
<td>203.80</td>
</tr>
<tr>
<td>Realized return (Jul 01-31)</td>
<td>-0.056</td>
<td>-0.065</td>
<td>0.019</td>
<td>-0.002</td>
<td>0.022</td>
<td>0.044</td>
<td>0.055</td>
<td>0.019</td>
<td>0.037</td>
<td>0.015</td>
</tr>
</tbody>
</table>

**Remark:** The estimated market impact parameters for January, 2015 in Table 1 show that the matrix $\Gamma - M$ is negative definite. When these parameters are applied for portfolio trading on Aug 03, the deleveraging model (21) is a concave maximization with three convex constraints and one nonconvex constraint. Generally, we cannot assume that the matrix $\Gamma - M$ is always negative definite.
Therefore, the initial (financial) leverage ratio is $\rho_0 = \frac{p_0^\top x_0 + L_0}{w_0} = \frac{1+1}{1} = 2.0$, see (4). Assuming each long asset contributes 100% to margin, i.e., $\eta_j = 1$, the initial margin ratio is, 

$$
\zeta_0 = \frac{-\sum_{j \in N} p_{1j} x_{1j}}{\sum_{j \in P} \eta_j p_{1j} x_{1j}} = 33.3\%.
$$

If the portfolio is not deleveraged, due to the market downturn predicted through return parameters, the latter two ratios are likely to worsen.\(^4\) Therefore, the effect of setting policy parameters, $\rho$ and $\zeta$ on leverage and margin, respectively, will be analyzed in determining an MV-optimal portfolio, $x_1^*(\rho, \zeta, s)$ for a given risk level $s > 0$, whose actual performance will be out-of-sample simulated during the (future) period Aug 04-31.

### 5.1. Model analysis and MV frontiers

Under the parameters in Table 1, the deleveraging model (21) for a given triple $(\rho, \zeta, s)$ is solved via the DCP method to determine the upper bound on expected portfolio net asset value, i.e.,

$$
F_u(\rho, \zeta, s) + e^{\tau_0} \phi \geq E[A_T(x_1^*(\rho, \zeta, s))].
$$

Starting with the infeasible portfolio $x_1^*(\rho, \zeta, s)$ that determines $F_u(\rho, \zeta, s)$, a sequence of feasible portfolios is generated using the procedure in Section 4, which converges to the most-improved feasible portfolio, denoted $y^K \equiv x_1^*(\rho, \zeta, s)$, labeled the near-optimal deleveraged portfolio, whose quality is given by $1 - \mathcal{E}$, see (49).

Table 2 provides evidence of model quality $(1 - \mathcal{E})$, solution convergence, and portfolio characteristics for leverage/margin policy parameters set at two different parameter settings: reducing the (financial) leverage from current 2.0 to $\rho = 1.1$, while changing margin ratio from current 33.3\% either by reducing to $\zeta = 20\%$ or by increasing to $\zeta = 60\%$. The initial portfolio in Table 1 has a standard deviation of $1,259.91$, while its upper bounding risk metric in (18) is $R_1(x_0) = 84,260.89$. We set the risk parameter at $s = 60,000$ for the sample model runs in Table 2, where 'after trading’ terms are defined as follow: long (portfolio) value = $p_1^\top x_1^+$, short (portfolio) value = $p_1^\top x_1^-$, where $p_i$ is given by (1), borrowed cash = $L_0 - K(x_1)$, and trading loss = $w_0 - p_1^\top x_1 + L_0 - K(x_1)$.

In the case of applying the standard MV optimization (MVO) model ignoring trading impact, one sets $\lambda = \gamma = 0$ model (21); however, the upper bounding risk constraint in (21) will be replaced by the exact portfolio variance constraint:

$$
x_1^\top P_0 V P_0 x_1 \leq (\sigma_P)^2, \quad (50)
$$

where $\sigma_P$ is set to either $649.22$ or $663.67$, corresponding to the two SDO model runs in Table 2. The MVO model with the risk constraint (50) is a pure convex quadratic optimization, which is solved using the standard software, IBM\textsuperscript{TM} CPLEX V12.7.1. Then, the optimal portfolio $x_1$ so-obtained is used to calculate the cash generated $K(x_1)$ by (2), and portfolio characteristics are computed based on what the price impact would have been under liquidity, i.e., using $p_i$ in (1). As

\(^4\)What would be the actual ratios if we were not to rebalance at all?
Figure 2  Deleveraging mean-variance efficient frontiers

Table 2 shows ignoring liquidity impact leads to undermining portfolio performance and increased trading costs, especially when deleveraging under stricter leverage and margin policies. For example, with \( \rho = 1.1 \) and \( \zeta = 0.6 \), ignoring liquidity costs during deleveraging results in trading costs increase by about 9\% for the MVO model; moreover, the end of the month expected net value of the optimal MVO portfolio also declines by nearly 9\% relative to the optimal SDO portfolio. Upon further testing, Figure 2 shows the efficient frontier of the monthly expected portfolio return and its standard deviation for the two models, SDO and MVO. Observe that ignoring market impact leads to over-estimated MV frontiers; however, when trading costs are incorporated ex-post to account for executing the optimal MVO portfolio, its performance becomes significantly-inferior. On the other hand, the optimal SDO portfolio incorporates liquidity costs ex-ante and its performance stays superior at all levels of portfolio risk. It must also be noted that the true MV frontier of the deleveraging problem is unknown because the SDO model in (21) only uses an upper bounding risk.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Illustration of deleveraged optimal portfolios on Aug 03, 2015</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>Parameters (( \rho, \zeta ))</td>
</tr>
<tr>
<td>Initial</td>
<td>(-)</td>
</tr>
<tr>
<td>SDO†</td>
<td>(1.1, 0.2)</td>
</tr>
<tr>
<td>SDO</td>
<td>(1.1, 0.6)</td>
</tr>
<tr>
<td>MVO‡</td>
<td>(1.1, 0.2)</td>
</tr>
<tr>
<td>MVO</td>
<td>(1.1, 0.6)</td>
</tr>
</tbody>
</table>

Portfolio weights: XLB XLE XLF XLI XLK XLP XLU XLV XLY

| Initial | - | -12.50% | -12.50% | -12.50% | -12.50% | 25.00% | 37.50% | 37.50% | 12.50% | 37.50% |
| SDO     | (1.1, 0.2) | -0.65% | -15.58% | 0.00% | 0.00% | 0.00% | 0.00% | 46.75% | 7.15% | 15.58% | 46.75% |
| SDO     | (1.1, 0.6) | -24.13% | -24.13% | 0.00% | -24.13% | 0.00% | 72.38% | 3.49% | 24.13% | 72.38% |
| MVO     | (1.1, 0.2) | -0.00% | -16.28% | 0.00% | 0.00% | 0.00% | 0.00% | 49.83% | 0.00% | 16.61% | 49.83% |
| MVO     | (1.1, 0.6) | -24.63% | -24.63% | 0.00% | -24.15% | 0.00% | 73.89% | 0.00% | 24.63% | 73.89% |

† SDO model is run by setting $s = 60,000$. Avg. CPU time is 0.065 seconds.
‡ MV optimization (MVO) ignoring trading impact, i.e., $\lambda = \gamma = 0$, and fixing $\sigma_p$. Avg. CPU time is 0.006 seconds.
* Expected net wealth on Aug 31 under trading costs for optimal portfolios of SDO or MVO models, see (16).
measure on portfolio variance (to alleviate the computational burden). This implies that the true frontier is never worse than what obtained under SDO by varying the portfolio risk metric, $s$.

5.2. Out-of-sample analysis

Portfolio net value random variable $A_T(x_1)$ at the end of rebalancing period is given by (14). Using $T \equiv \text{Aug 31}$, the realized value of portfolio net wealth, denoted by $\hat{A}_T(x_1)$ and referred to as the out-of-sample (OOS) portfolio value, is determined using the actual asset returns of Aug 31, with discrete simulation of trade execution during Aug 03.

For discrete simulation, trading is performed every 5-mins (which corresponds to the impact parameter estimation process), assuming the execution at the VWAP price, $p_{t}^{vw}(j)$, during each 5-min interval on day 1, Aug 03. As an example, suppose the optimization determined $x_{1j} - x_{0j} = 10,000$ shares for asset $j$, a net increase in the position (for cover-buying in a short position). Then, under the constant trading strategy, in simulation with $S = 78$ time steps, $10,000/78 \approx 128$ shares are bought every 5-mins at $p_{t}^{vw}(j)$ at every time interval $t = 1, \ldots, S$. The total cost of executing this position is then $\$ \sum_{t=1}^{S} 128 p_{t}^{vw}(j)$. Then, the total cash generated in portfolio trading is:

$$\hat{K}(x_1) = - \sum_{j=1}^{n} \sum_{t=1}^{S} \left[ \frac{x_{1j} - x_{0j}}{S} \right] p_{t}^{vw}(j).$$  \hspace{1cm} (51)

Denoting the actual asset prices on day $T \equiv \text{Aug 31}$ by $p_{T}^{act}$, the OOS portfolio value is:

$$\hat{A}_T(x_1) = (p_{T}^{act})^\top x_1 + e^{r_0} [\hat{K}(x_1) - L_0].$$  \hspace{1cm} (52)
6. Concluding Remarks

This paper presents a new approach to portfolio deleveraging incorporating margin and leverage constraints under price impact due to market illiquidity. The resulting model is a difficult non-convex optimization model involving quadratic objective and constraint functions. We proposed a novel dual cutting plane methodology to solve the Lagrangian upper bounding problem efficiently. Moreover, a lower bounding feasible portfolio generation algorithm is proposed so that the optimality of the deleveraged portfolio so-obtained can be ascertained.

Our deleveraging application using ETF assets demonstrates the validity and the efficacy of the approach we have taken in this paper. Furthermore, the portfolio analysis discussed in this paper provides insights into how policy parameters on portfolio margin/leverage must be considered when future uncertainties demand portfolio rebalancing to avoid portfolio catastrophe of the kind witnessed in past economic events.

References


Finding centroid in DCP method and Market impact estimation

**EC.1. Appendix A: Finding the centroid of** \( \Theta_t \)

We employ the COP (Cut-Off Polyhedron) method in Nakagawa et al. (1984) to determine the centroid of a polyhedron defined by a finite set of hyperplanes. We present the basic elements briefly here by adapting and modifying to our dual cutting plane context.

The initial polyhedron in (27), referred to as \( \Theta_0 \), is formed by \( m + 1 \) (facet-inducing) hyperplanes and \( m + 1 \) vertices, given by \( m \) the elementary vectors \( e_1, \ldots, e_m \) and the origin \( 0 \). After \( t \) iterations of the DCP technique, the resulting polyhedron \( \Theta_t \) in (48) is determined by \( K_t \) hyperplanes (after removing the non-facet-inducing and redundant hyperplanes from the total of \( m + 1 + t \)), and its set of extreme points is denoted by \( V_t = \{ v^r : r = 1, \ldots, R_t \} \). Note that both \( K_t \) and \( R_t \) are non-monotonic in \( t \), and they have a minimum value of \( m + 1 \). For the remainder of this section, we shall drop the iteration count sub (super-)script \( t \) for ease of exposition.

Let \( Y \) be a logical (0-1) matrix that indicates which of the \( K \) hyperplanes are connected by which of the vertices in \( V \) to form the matrix \( Y \in \mathbb{R}^{R \times K} \), where its \((r,k)\)th element:

\[
[Y]_{r,k} = \begin{cases} 
1 & \text{if a vertex } v^r \text{ is on the } k^{th} \text{ hyperplane} \\
0 & \text{otherwise.}
\end{cases}
\]

For the initialization step with \( \Theta_0 \), the logical matrix \( Y_0 = U - I \in \mathbb{R}^{(m+1) \times (m+1)} \), where for the square matrix \( U \) all elements are unity, i.e., \([U]_{i,j} = 1\), and \( I \) is the \((m + 1)\)-dimensional identity matrix.

Note that \( C(\Theta) = \frac{1}{\pi} \sum_{r=1}^{R} v^r \), and thus, the dual cutting plane satisfies \( \alpha^\top C(\Theta) = \beta \). For the polyhedron updated by the dual cut, i.e., \( \Theta \cap \{ \alpha^\top \theta \geq \beta \} \), the updated set of vertices is required. Then, the following operations are performed.

Step 1: Determine the index sets of feasible and infeasible vertices of \( \Theta \):

\[
R^- = \{ r : \alpha^\top v^r < \beta \},
R^0 = \{ r : \alpha^\top v^r = \beta \}, \text{ and}
R^+ = \{ r : \alpha^\top v^r > \beta \},
\]

where \( R^- \) is the set of indices for infeasible vertices under the dual cut, \( R^0 \) is the set of indices for vertices that are on the dual cut, and \( R^+ \) is the set of indices for vertices that are strictly feasible under the dual cut. Note that \( R^+ \neq \emptyset \) since \( C(\Theta) \) is in the relative interior of \( \Theta \) and \( \alpha^\top C(\Theta) = \beta \).

Step 2: Find planes that will form a new \( \Theta \).

\[
K^+ = \left\{ k : \sum_{r \in R^+} [Y]_{r,k} > 0, \ k = 1, \ldots, K \right\}
\]
Step 3: Find new vertices of a new $\Theta$. The new vertices are generated where the dual cut meets edges made of a vertex in $R^+$ and a vertex in $R^-$. Thus, for all pairs of $r^+ \in R^+$ and $r^- \in R^-$, do the following sub-steps:

- Step 3.1: Get a 0-1 vector $t \in \mathbb{R}^{|K^+|}$ that indicates whether the a plane formed by $v^{r^+}$ and $v^{r^-}$:
  \[ t = [Y]_{r^+,k} \cdot [Y]_{r^-,k} \text{ for } k \in K^+ \]

- Step 3.2: If $\sum_{j=1}^{|t|} t_j = m - 1$, then, since $v^{r^+}$ and $v^{r^-}$ are on the same edge, do the following two sub-steps:
  - Update $R \leftarrow R + 1$, $R^0 \leftarrow R^0 \cup \{ R \}$, and $[Y]_{r,k} = t^T$ for $k \in K^+$.
  - Find a new vertex $v^R = v^{r^+} + \sigma(v^{r^-} - v^{r^+})$, where the step size $\sigma$ is
    \[ \sigma = \left( \beta - \alpha^T v^{r^+} \right) / \left( \alpha^T [v^{r^-} - v^{r^+}] \right) \]

Step 4: Add a new column on $Y$ for the hyperplane of the dual cut.

- Set $Y_{r,K+1} = 1$ for $r \in R^0$ and $Y_{r,K+1} = 0$ for $r \in R^+$

Step 5: Reconstruct $Y$ and $V$ for the new polyhedron $\Theta$

- $R = |R^+| + |R^0|$, $K = |K^+| + 1$, $V = \{ v^r : r \in R^+ \cup R^0 \}$
- $Y = [Y]_{r,k}$ for $r \in R^+ \cup R^0$ and $k \in K^+ \cup \{ K \}$

Step 6: Get the centroid $C(\Theta) = \frac{1}{R} \sum_{r=1}^{R} v^r$.

EC.2. Appendix B: Estimating Market Impact Parameters

Market impact parameters are estimated using TAQ (Millisecond Trade and Quote) data from NYSE. The trade (execution) transactions data for a given day (for given stock or ETF) are aggregated into five-minute intervals during regular trading time. Then, the total of 390 minutes per day (from 09:30-16:00 hrs) is divided into $T = 78$ five-minute intervals, for each trading day in the sample of $N$ number of days. For a given day $d$, at some time $t$, the price of a (generic) asset is denoted by $p_{t,d}$. Hence, the open price for the day is $p_{0,d}$. By classifying each trade using the tick rule in Asquith et al. (2009), the following trade statistics are calculated for each five-minute interval (denoted $t$) in the day $d$:

1. Total trade volume: $v_{t,d}$ (shares)
2. Net trade volume: $\hat{v}_{t,d}$

$^5$The net volume of a stock is a consolidated total of the positive and negative movements of the security over the period, i.e., up-tick volume minus its down-tick volume in the 5-min interval. A positive (negative) net volume indicates greater upward (downward) movement associated with net ‘buying’ ('selling') in the security over the five minutes.
3. Dollar trading volume: $s_{t,d}^6$

4. Open and close prices (in the 5-min interval): $p_{t,d}^o$ and $p_{t,d}^c$

5. Given an interval $(t,d)$, let there be $n$ distinct trades. The price change of two consecutive trades is denoted $δp_i$, if greater than $0.005$, otherwise, $0$. The sum of absolute values of price changes in the 5-min interval: $Δp(t,d) = \sum_{i=1}^n |δp_i|$

The total net volume traded in a day until time $t$ is $\sum_{\tau=1}^t \hat{v}_{\tau,d}$. This leads to the permanent price change by the end of period $t$ given by $(p_{t,d}^c - p_{0,d})$, the price differential at the end (close) of the time period and the price at the beginning of the day. On the other hand, the temporary impact is due to trading at higher rates causing a temporary lack of liquidity to absorb the required trading rate.

Given a 5-min time period, the temporary price effect is measured by the product of the level of illiquidity in the period, $I_{t,d}^L$, and the (effective) trade price against illiquidity, that is, $I_{t,d}^L \times p_{t,d}^{vw}$, where volume-weighted average price (VWAP) is used for the effective price. We employ the most widely liquidity measure developed by Amihud (2002), and define:

$$I_{t,d}^L = \frac{Δp(t,d)}{s_{t,d}}$$

as the measure of illiquidity during a 5-min interval. Then, the temporary effect is

$$I_{t,d}^L \times p_{t,d}^{vw} = \frac{Δp(t,d)}{s_{t,d}} \times \frac{s_{t,d}}{v_{t,d}} = \frac{Δp(t,d)}{v_{t,d}}.$$ \hspace{1cm} (EC.2)

Combining the permanent and temporary price effects, we estimate the model:

$$(p_{t,d}^c - p_{0,d}) + Δp(t,d) = γ \sum_{\tau=1}^t \hat{v}_{\tau,d} + λv_{t,d} + ε_{t,d}, \quad t = 1, \ldots, T, \quad d = 1, \ldots, N.$$ \hspace{1cm} (EC.3)

References


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$^6$The dollar volume is computed by multiplying each trade size by the execution price of the trade, and summing up over all trades in the 5-min interval.