Benchmarking Asset Allocation

Abstract

This paper shows how to use Roll’s efficient set mathematics to construct the efficient frontier and the optimal Sharpe tangency portfolio in a simple bond, money and equity market framework. The application to markets rather than individual stocks limits the impact of some of the well-known problems of the sensitivity of the output to the inputs, particularly the expected rates of return. The objective is to show how this can benchmark forecasts and constrain the exercise of professional judgment to the basic asset allocation decision. The specific data used is a forecast provided by a prominent actuarial advisory firm operating in the current period of unconventional monetary policy. The result is to highlight basic relationships and position the optimal Sharpe portfolio relative to the standard pension fund allocation and the risk parity portfolio.
Benchmarking Asset Allocation

The basic asset allocation decision is that between common equities and bonds. In an influential paper, Brinson et al (1986) analyzed the performance of 91 pension funds over the period 1974-1983. By substituting actual benchmark returns for each pension fund’s normal asset allocation, they concluded that asset allocation explained over 90% of the variation of each pension fund’s return. Subsequently Brinson et al (1991) confirmed their earlier result, which has been confirmed by others, including, for example, Ibbotson and Kaplan (2000), and Hlawitschka and Tucker (2006). However, as Jacobsen and Biwer (2011) pointed out, the importance of asset allocation depends critically on the realized equity risk premium, that is, the premium actually earned on equities over bonds. If the realized equity risk premium has been very low or non-existent then asset allocation over that particular period is less important than if it had been very large.1

The problem that expectations are seldom realized is endemic to empirical work in finance. It is difficult to solve: too short a period increases the risk of the non-realization of expectations, whereas a longer period increases the risk of structural changes impacting the results.2 However, as well as looking at actual equity and bond market returns, this paper also examines asset allocation on a going forward basis using expected returns. It does this by using the efficient set paradigm synthesised in an appendix by Roll (1977)3 combined with a set of expected returns developed by a well known actuarial firm specialising in capital market advisory support, especially for pension funds. As is well known, the use of the efficient set, while being fundamental to portfolio theory, is very sensitive to the choice of parameters, particularly expected rates of return. However, when looked at from the perspective of asset allocation, that is, in terms of the money, bond and equity markets, this criticism is not as

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1 In the same way, back-testing over recent time periods simply reflects the financial circumstances of that time period, not what was expected.

2 See Marmer (19660 for an interesting perspective on structural changes.

3 Ross attributes many of the results to others, notably Black (1972).
important as it is for individual securities where the derivation of expected returns is more
difficult.

Further, the efficient set algorithm provides a very simple framework for sensitivity analysis
with respect to the required parameters when dealing with key markets rather than securities.
This makes the framework ideal for what if analysis and benchmarking both judgment and
historic and future portfolio performance.

II Efficient portfolio allocation

Since Markowitz (1954) and Roy’s (1952) development of modern portfolio theory, finance has
had a well-developed framework for analyzing the risk return trade-off involved in portfolio
construction and the formation of the efficient set. However, Markowitz’s critical line algorithm
focused on short and long positions in a variety of securities and technical issues in the
formation of the efficient set. In contrast, Roll’s development is particularly useful when only a
limited number of markets are involved as well as for understanding how and why portfolio
allocation varies with changes in the input parameters. In this way, Roll’s efficient set
mathematics is very useful for sensitivity analysis.

The efficient set is derived by choosing the weights \( W \) to minimize the portfolio variance \( V \)
for a given expected return \( E \), subject to the condition of being fully invested. Mathematically
the problem is:

\[
\min_{W} \quad WSW^T + 2 \cdot \lambda_1 [WR - E] + 2 \cdot \lambda_2 [We - 1]
\]

\( W \) is the 1 by \( n \) vector of security holdings, \( S \) is the \( n \) by \( n \) variance-covariance matrix, \( E \) is the
portfolio’s target rate return, \( e \) is an \( n \) by 1 vector of ones, such that the security holdings sum
to 1 and \( \lambda \) subscripted 1 and 2 are the Lagrangian multipliers for the two constraints. \( T \)
indicates transpose.

Solving gives the following set of first order conditions,

\[
SW^T + \lambda_1 R + \lambda_2 e = 0
\]
Inverting the variance-covariance matrix to solve for the optimal weights gives

\[
W^T = S^{-1}[R e] \lambda
\]

Following Roll $\lambda$ is a 2 by 1 vector of the two (negative) Lagrangian multipliers and $[R e]$ is an $n$ by 2 matrix formed by combining the two column vectors of expected returns and ones.

The simplicity of Roll’s solution for the efficient set algorithm lies in pre multiplying both sides of the solution for the optimal weights by the transpose of the $[R e]$ matrix, where grouping terms on the right hand side Roll defined a new matrix $A$ as,

\[
A = [R e]^T S^{-1} [R e]
\]

The important result for the $A$ matrix is that it is always a symmetric 2 by 2 matrix regardless of the number of securities and only involves the expected returns and variance-covariance matrix. Roll denoted the 1, 1 entry in $A$ as “a”, the 2,2 entry as “c” and the off diagonal entries as “b”; a convention followed here.

Inverting $A$ solves for the Lagrangian multipliers,

\[
\lambda = A^{-1} [R e]^T W
\]

However, $[R e]^T W$ simply reflects the values of the two constraints, that is, the target rate of return and 1, so can be expressed as the 2 by 1 vector of $E$ and 1. In this way the optimal portfolio weights for any given target rate of return are equation (1) below,

\[
W^T = S^{-1} [R e] A^{-1} \begin{bmatrix} E \\ 1 \end{bmatrix}
\]

This solution involves no investor preferences, simply the expected rates of return and the variance-covariance matrix. It is thus extremely general and while it may look complicated, it simply involves a few cells in a spreadsheet.

With these optimal portfolio weights, Roll derived a number of important results. The first is for the portfolio variance, which is $V = W^T S W$. If we use the solution for the optimal weights and the components of the $A$ matrix we get equation (2) below
The portfolio variance is then a parabola in terms of the expected portfolio return \((E)\) with the specific form determined by the parameters \(a\), \(b\) and \(c\) from the matrix \(A\).\(^4\) As the target expected rate of return increases so too does the variance, but at a decreasing rate. The combination of \(V\) and \(E\) then maps out the entire efficient frontier.

The second important result is that for any efficient portfolio, call it “\(U\)”, there is an orthogonal portfolio “\(Z\)”, where there is no correlation in the rates of return between the two. If risk is the standard deviation of the portfolio’s return the tangent line from the expected return on portfolio \(Z\) is tangent to the efficient frontier at \(U\).\(^5\) This result was earlier derived by Black(1972), whose interest was in a capital market equilibrium without a risk free asset. Black showed that in such an equilibrium the market portfolio \((M)\) is still optimal. In this case, a tangent line to \(M\) has an intercept equal to the expected return to its orthogonal portfolio, which Black termed the minimum variance, zero beta portfolio. In this way, Black derived his celebrated pricing model, which is identical to the capital asset pricing model, except for the substitution of the expected return on the zero beta portfolio for the risk free rate.

Roll showed that it is always a property of the efficient set that for any efficient portfolio the tangent line from its expected rate of return touches the efficient frontier at its orthogonal portfolio. Further, there is a relationship between the expected rates of return on these two portfolios as represented in equation (3) below:

\[
R_U = \frac{a - bR_Z}{b - cR_Z}
\]

For our purposes, this means that we can choose \(Z\) as a proxy for the risk free rate and then find its orthogonal portfolio \(U\). By definition, this portfolio has the maximum Sharpe ratio since it is

\(^4\) This is the equation of the efficient frontier, of which the efficient set is a sub set.
\(^5\) When risk is measured as the variance of the portfolio’s rate of return, the line between \(U\) and \(Z\) cuts through the expected return on the minimum variance portfolio.
the tangent portfolio with respect to that proxy for the risk free rate and the specific values sued to form the efficient set.

This provides a simple tractable model for analyzing asset allocation, since we only have two risky markets, the equity market and the bond market, and one risk free “money” market. Here we note that equity is a levered security, since corporations have debt outstanding. As a result, the true market portfolio, at the very least, involves a portfolio involving both equities and corporate bonds. In this case, the solution algorithm is to:

1) First, choose an appropriate proxy for the risk free rate,
2) Second, use the return on this risk free asset as the expected return on the risky portfolio (Z) that should lie on the inefficient part of the efficient frontier.
3) Third, calculate the expected return on Z’s orthogonal portfolio, U, using Roll’s equation (3) above,
4) Fourth, determine the optimal weights for portfolio U, which as the tangent portfolio by definition has the maximum Sharpe ratio using equation (1) above.
5) Finally, we can determine the tangent portfolio’s variance and standard deviation using equation (2) above, and its Sharpe ratio.

The above algorithm holds regardless of whether or not the market is “in equilibrium” since it only depends on the construction of the efficient set. If we impose equilibrium conditions then U is the market portfolio. However, the concern here is only for asset allocation across the three major markets, not whether or not the maximum Sharpe ratio portfolio is in a broader sense the true market portfolio.

To illustrate the mechanics of the algorithm as well as some of the problems with using historic data, the historic average annual returns since 1934 on the Canadian equity, bond and Treasury bill markets are in in Table 1 along with the consumer price index. All that is missing is the correlation between the equity and bond markets, which over the entire period is -0.043 or

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6 The analysis is easily extended to other key markets, such as commodities and alternative investments.
7 Canadian Treasury Bill data starts in 1934 when the Canadian government initiated a Treasury bill auction. The data mirrors that of the US markets and is used for illustrative purposes.
essentially zero. The simple average, or arithmetic, rate of return on Canadian equities is
11.37%, while the standard deviation was been 16.58%. For bonds, the values are 6.54% for the
average return and 9.12% for the standard deviation of that return.

If these historic average values are substituted into Roll’s $A$ matrix we get

$$A = \begin{bmatrix} 0.986 & 0.120 \\ 0.120 & 0.016 \end{bmatrix}$$

With the average Treasury Bill return as the risk free rate of 4.50% the maximum Sharpe
portfolio has a return of 8.98% calculated using equation (2) as,

$$0.0898 = \frac{0.986 - 0.120 \times 0.045}{0.12 - 0.016 \times 0.045}$$

If this expected return and 1 are both substituted into the equation for the efficient portfolio
weights, equation (1), we get

$$\begin{bmatrix} 0.504 \\ 0.496 \end{bmatrix} = \begin{bmatrix} 0.207 & -1.36 \\ -0.207 & 2.36 \end{bmatrix} \begin{bmatrix} 0.0898 \\ 1 \end{bmatrix}$$

Using the historic values since 1934, we find that the tangent portfolio that maximized the
Sharpe ratio was 50.4% equities and 49.6% bonds, essentially the standard 50:50 portfolio of
many pension funds. Finally, using the equation (3) for the variance, the maximum Sharpe
portfolio has a standard deviation of 0.095 and a Sharpe ratio of 0.471. The Sharpe ratio for the
equity market alone is 0.414 and the bond market 0.224. As we would expect with just two
markets, 100% holding in either market is inefficient when there is little correlation between
the two.

These results are in Table 2 along with the results for two “competing” asset allocation models:
the minimum variance portfolio, and an allocation of 60% to equities and 40% to bonds, which
is also a common pension fund allocation. The minimum variance portfolio is representative of
the return on a low volatility portfolio,\(^8\) which in this special case of close to zero correlation is

also the return from a risk parity strategy.\footnote{9} Roll pointed out that the return on the minimum variance portfolio is simply b/c from the A matrix, where its composition can then be found in the same way as for the maximum Sharpe portfolio. With the historic data, the return on the minimum variance portfolio is 7.66% with a Sharpe ratio of 0.396. The optimal weights are 23.1% equities and 76.9% bonds. With the historic data, the standard 60:40 allocation has an average return of 9.04% with a Sharpe ratio essentially the same as the optimal Sharpe ratio of 0.466.

Of importance is that even with these historic data over the long period 1934-2016 different common strategies produce very similar Sharpe ratios. In this case, the optimal Sharpe strategy is very close to both the 50:50 and 60:40 pension fund strategies, while even the Sharpe ratios for the minimum variance and risk parity portfolios have not been too dissimilar. Further, Table 1 shows that the distribution of returns may not be normal as the average value differs from the median value for all four series. For the equity series, the difference of the median from the average of 0.65% is not significantly different from zero (T statistic -0.354), but it is still slightly left skewed indicating that some very low returns drag down the average. In contrast, for the CPI and fixed income series they are all right-skewed as the average exceeds the median values. Moreover the differences are all significant at 0.98% for the CPI (T statistic 2.63), 0.84% for T Bills (T statistic 1.81) and 2.28% for bonds (T statistic 2.29).

For the CPI and fixed income returns, the assumption that they are random returns from a constant distribution is questionable. As a result, the assumption that past average returns are good predictors of future returns is suspect. One possibility may be the change in inflation over the period 1934-2016. In Table 3 are the descriptive statistics for the real return series, where the real return is simply one plus the annual return divided by one plus the realized inflation rate minus one.

For equities, there is no substantial change: the median still exceeds the average, but the difference of 0.65% is not statistically significant (T statistic -0.364). In contrast, for the bond

\footnote{9 This equalizes risk using the variances of equity and bond market returns.} 
\footnote{10 See Baitinger et al (2017) and Anderson (2012) for recent examples.}
series there is a big difference as the median now exceeds the average, rather than the reverse, and the difference of -0.20% is not significant (T statistic -0.188). However, the T Bill series now has a different interpretation. Like the real bond series, the median now exceeds the average, but unlike the bond series, the difference of 0.65% is marginally significant (T statistic -1.617).\textsuperscript{11}

The fact that the average Treasury bill real return may not be a good predictor of future returns should not be surprising. In Figure 1 is a graph of the annual average values for CPI inflation, and Treasury bill and bond yields. There are three distinct periods: 1934-1955, 1956-1981 and 1982 to 2016. From 1934 to 1955, there was almost no relationship between inflation and the annual yields to maturity on either bonds or Treasury Bills. It was not until the Bank of Canada made a series of reforms in 1953-54 that a functioning money market in Canada even existed.\textsuperscript{12} As Watts (1974) explains, it was not until the mid-1950s “that the Bank started to pursue an active monetary policy” prior to then it had neither the tools nor the inclination. The period from 1956 until 1981, was one of accommodating monetary policy with rising inflation that peaked at an annual rate of 12.12% in 1981. Long Canada bond yields (Canadas) increased during this period from 3.14% in 1955 to a peak of 15.22% in 1981 causing losses in bond portfolios with actual returns significantly less than expected. The impact of monetary policy is most obvious in the Treasury bill yield with the increase from 1.63% to 18.99% as the Bank of Canada enacted restrictive monetary policy to reduce the high rate of inflation. Post 1981 inflation has come down and at a slightly delayed pace so too have Long Canada bond yields and Treasury bill yields.

The upshot of the data in Figure 1 is that the changed impact of monetary policy clearly affects inflation and the returns on fixed income securities. It is heroic to assume that monetary policy in the future will retrace its experience since 1934. Further, there is a substantial recent change caused by unconventional monetary policy or quantitative easing that has no previous history. Santor and Suchanek (2016) report that at the end of 2015 the US. Federal Reserve had bought

\textsuperscript{11} Using these real returns the equity allocation of the maximum Sharpe strategy increases slightly to 52%, but the other results are essentially the same.

\textsuperscript{12} See Watts (1974) prior to the reforms of 1953-54 there was no liquid secondary market trading in money market securities in Canada.
$4.2 trillion in bonds amounting to 18% of the US. Treasury bond market and 28% of the US agency mortgage backed security market. In the UK the Bank of England had bought 32% of the government bond market; the European Central Bank 21% of the Euro government debt market and the Bank of Japan 36% of the Japanese government bond market.

In total, central bank bond purchases amount to almost C$13 trillion; a level, which has clearly affected long-term market interest rates: the Japanese buying alone amounts to 80% of Japanese GDP. If this C$13 trillion had instead, been placed in private markets, the level of long-term interest rates would have been substantially higher. In turn, this means that recent bond market gains would instead have been losses. In 2016, long Canada bond yields were only 1.80%, which is less than the Bank of Canada’s target inflation rate of 2.0%. Of importance is that the Government of Canada is one of very few AAA rated issuers, so this yield to maturity of 1.8% is an expected rate of return. To ignore the current bond market conditions in asset allocation is a mistake. Instead, a current forecast is needed which includes current monetary policy.

III: Reasonable Values

Past performance constrains and guides judgement. However, the return on fixed income securities depends on the actions of central banks, which the previous section has shown have exhibited structural changes. Consequently, a current forecast taking into account current monetary policy is useful. For this purpose, Table 3 includes the basic forecast data from an AON-Hewitt ten year 2016 forecast of the Canadian market. The forecast includes both the ten year forecast compound return as well as the simple arithmetic average annual return. At that time, the forecast average return on long Canada bonds was 2.10%, similar to the 1.8% yield to maturity for 2016. Overall, the arithmetic annual average return on bonds was forecast to be 2.50% while that for Canadian equities was 8.30%, US equities 7.60% and global equities in between at 7.80%. These forecast expected returns were significantly lower than the historic

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13 With lower rated debt, the yield to maturity is a promised and not an expected rate of return due to default risk.
14 This comes from a capital market forecast provided in a public utility hearing in Newfoundland CA-NP-269 Attachment B.
average returns reflecting the lower average CPI inflation forecast. In contrast, the standard deviations of the annual returns of 17% for equities and 9.90% for bonds are very similar to the historic averages since 1934.\textsuperscript{15} AON-Hewitt forecast an equity-bond market correlation of 0.1, which was slightly higher than the historic average, but still not materially different from zero.

If AON-Hewitt’s forecast values are used in Roll’s equations for the efficient set, we get the results in Table 5. The optimal Sharpe portfolio now has a 91.6% equity allocation with an expected return of 7.8% and a Sharpe ratio of 0.377. Both the expected return on the optimal Sharpe portfolio and its Sharpe ratio are significantly lower than that produced from using historic average values regardless of whether they are nominal or real. The reason for the high equity allocation is the relative pessimism of AON-Hewitt’s forecast, reflecting the prevailing view at the time of a “new normal” in forecast returns.

The Sharpe ratio implicit in AON-Hewitt’s equity market forecast is 0.376, marginally lower than the historic average of 0.414. However, it is the government bond market Sharpe ratio of 0.020, which is the outlier, since it is much lower than the historic average of 0.224. Even the Sharpe ratio for the overall bond market, including corporate and securitized bonds, is not that much higher at 0.056. Both these Sharpe ratios reflect the impact of unconventional monetary policy and the very low level of bond yields at the time. Reflecting the low bond market Sharpe ratio, the minimum variance portfolio has a Sharpe ratio of 0.207 with a slightly higher equity market weight. In contrast, the 60:40 portfolio has a Sharpe ratio of 0.358 reflecting the lower Sharpe ratios for both the equity and bond markets.

The use of forecast data indicates that unconventional monetary policy clearly has its desired impact of subsidizing borrowing to finance the purchase of real assets. For asset allocation, this means the optimal Sharpe portfolio has an increased equity allocation. However, as long as equity and bond markets have very little correlation, some allocation to the bond market is still optimal. This leads to the question, what are the critical values in deriving the optimal equity allocation?

\textsuperscript{15} AON-Hewitt estimated them as the historic average over the period 1987-2014, whether such an approach is valid for bonds given the low level of expected returns is debateable.
The Sharpe ratio is the slope of the capital market line that trades off expected return against portfolio risk, where risk is the standard deviation of the portfolio return. Given the relative stability of the standard deviation of returns, the lower Sharpe ratios for both equity and bond markets reflect lower expected risk premiums. This leads to two critical values: the equity risk premium over bonds, which affects the efficient set formation and the proxy for the risk free rate that generates the tangency portfolio. The Treasury bill yield will be used as the proxy for this risk free rate.

Using arithmetic annual returns the historic equity risk premium (ERP) over bonds from Table 1 has been 4.83%.\(^{16}\) Using the AON-Hewitt forecast data, Table 6 gives the optimal equity allocation for a risk free rate of 0%, 1%, 2% and 3%, as well as the expected return on the minimum variance portfolio, as the ERP increases from 3% to 9%. For example, at a 3% ERP the expected equity return is 5.1% and the optimal equity allocation 45.2% at a 0% Treasury bill return. The optimal Sharpe portfolio has an expected return of 3.50% and a Sharpe ratio of 0.367, both well below historic averages. The expected return on the minimum variance portfolio is 2.9% with a 25% equity allocation and a Sharpe ratio of 0.334. The 60:40 portfolio has an expected return of 3.9% and a Sharpe ratio of 0.356.

As the ERP increases, it shifts the efficient frontier up and normally increases both the equity allocation and the expected return on the minimum variance portfolio.\(^{17}\) For example, at a 9% ERP, the equity allocation increases to 64.2% for the maximum Sharpe portfolio, which has a Sharpe ratio of 0.687 and an expected return of 7.9%. Similarly, the expected return on the minimum variance portfolio increases to 4.4% with a Sharpe ratio of 0.512. Since the risk of the equity and bond markets has not changed, the equity allocation for the minimum variance portfolio is constant at 25.32%.

Of importance is that the 3% to 9% ERP range encompasses what most would regard as extreme values since the implied Sharpe ratios at these ERP levels are both significantly different from historic values. However, despite this wide range for the ERP the equity

\(^{16}\) Using the full Canadian Institute of Actuaries data back to 1924 it is 4.85%.
\(^{17}\) The standard deviation of the minimum variance portfolio is constant at 8.56%.
allocation only varies from 45.2% to 64.2%. In each case, the lack of correlation between the equity and bond market returns serves to maintain a significant bond market allocation even at a very low bond market Sharpe ratio.

At a 1% risk free rate, the efficient set itself does not change: all that changes is the tangent to the efficient frontier generating the optimal Sharpe ratio. In this case, the bond market Sharpe ratio decreases to 0.111 and the equity market Sharpe ratio with a 3% ERP to 0.241. As a result, the optimal Sharpe portfolio has a Sharpe ratio of 0.266 and an equity allocation of 55.83%, up from the 45.16% that was optimal with a 0% risk free rate. Of importance is that increasing the proxy for the risk free rate, increases the equity allocation and reduces the bond market allocation. The reason is simply that the higher intercept causes a flatter tangent line to the efficient frontier. Another way of saying this is that whereas the bond market Sharpe ratio halves from 0.211 to 0.111, the equity market Sharpe ratio only drops from 0.3 to 0.212 so it is the bond market that bears the brunt of the higher risk free rate. At this equity allocation, the optimal Sharpe ratio is very similar to the standard 60:40 portfolio, which has a Sharpe ratio of 0.265

As before as the ERP increases, the equity market’s Sharpe ratio increases making it more attractive than the bond market. At an ERP of 9%, the equity market Sharpe ratio is 0.594 and the equity allocation increases to 75.7%. With a 1% risk free rate, the range for the equity allocation as the ERP ranges between 3% and 9% is 55.8% to 75.7%. For a 2% Treasury bill yield, these changes are even more apparent as the bond market Sharpe ratio shrinks to 0.01 and the equity market Sharpe ratio to 0.182; again it is the bond market that bears the brunt of the higher risk free rate. With the 2% risk free rate, the equity allocation is now in a range 91.3% to 98.6% as the ERP varies between 3% and 9%.18

At a 3% risk free rate, we have the most interesting case since the tangent portfolio now has an equity allocation of -379.2%! The reason being that the risk free rate exceeds the expected

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18 As the ERP increases, the equity allocation gets closer to 100% as the diversification gains from the bond market are increasingly offset by the loss in expected return.
return on the minimum variance portfolio of 2.9%. As a result, the return on the minimum variance portfolio with a 3% expected return is no longer on the inefficient part of the efficient set; instead, it is on the efficient part. Consequently, its orthogonal portfolio with the tangent line to 3% is on the inefficient part of the efficient set leading to a negative equity allocation.

As the ERP increases, so too does the expected return on the minimum variance portfolio, so the optimal Sharpe portfolio returns to the efficient part of the efficient set. For example, at a 4% ERP the expected return on the minimum variance portfolio is 3.10%, which is still very close to the 3% proxy for the risk free rate. As a result, the optimal Sharpe portfolio has an equity allocation of 694.7% with a massive short position in the bond market. This is because the Sharpe ratio for the bond market at a 3% proxy for the risk free rate is -0.091, which implies an arbitrage opportunity: the risky bond market offers a return of 2.1%, which is below the 3% return for the proxy for the risk free rate. As the ERP increases, you would expect the equity market allocation to increase. However, with a negative Sharpe ratio for the bond market this implies increasing risk due to the short position in the bond market. Instead the equity allocation is reduced as the ERP increases, so that with a 3% proxy for the risk free rate the range for the equity allocation as the ERP varies from 3% to 9% is -379.4% to 148.7%.

The relationship between the risk free rate and the expected return on the minimum variance portfolio highlights the fundamental problem of the efficient set. This is that both the proxy for the risk free rate, the Treasury bill yield, and the expected return on the long bond are borrowing costs that only differ according to their maturity. However, the efficient set is derived from the optimization over an undefined single period. As Black (1972) pointed out there may be no risk free asset, which is another way of saying that the Treasury bill yield may be appropriate for 90 days in which case the expected return from the long bond and the equity market should both be for a three month horizon. Conversely, if the time horizon is ten years, the return on the three-month Treasury bill has to be the return from rolling over

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19 When the risk free rate equals the return on the minimum variance portfolio there is no tangent line to the efficient frontier.
20 As the ERP increases, as before the equity allocation converges to 100%.
treasury bills for ten years. In this case, the return is no longer risk free and instead the return should be that on a ten-year discount note extracted from government bonds.

In the AON-Hewitt forecast of ten-year returns, they estimated the compound return over that period, which is also in Table 4. These ten-year compound returns are simply the annualized arithmetic returns over a single ten-year period. To be consistent with this forecast we take the compound return from rolling over Treasury bills for ten years of 1.90% as the proxy for the risk free rate, since the forecast volatility is only 1.0%. However, AON-Hewitt’s forecast for the long Canada bond is only 1.60%, which is inconsistent with this Treasury bill forecast, as well as the forecast 1.90% inflation rate. If these values are correct, the implication is of a negative Sharpe ratio for the bond market of -.03 and an equity allocation of 120% despite the low equity market Sharpe ratio of 0.306. The lower bond return in turn simply reflects their expectation that long Canada bond yields will increase as the yield curve normalizes. In this respects there is nothing wrong about such a forecast it simply cannot be an equilibrium forecast.

Conclusions

The efficient set mathematics as derived by Roll are a very useful check on the consistency of forecast data, particularly when dealing with relatively few markets, such as the basic equity-bond market allocation. In such a framework it is easy to see the importance of asset allocation both historically and the unique problems generated by unconventional monetary policy.

Historically, we can see that standard equity allocations of 50% to 60% for pension funds are consistent with historic data in the sense that their implied Sharpe ratios are very similar to that for the optimal Sharpe portfolio. Further, the minimum variance portfolio also has a reasonable Sharpe ratio, but this mainly reflects the historic high Sharpe ratio for the bond market of 0.224.

Using recent forecast data poses serious problems due to the impact of unconventional monetary policy. This is because forecast long run returns in the bond market constrain the proxy for the risk free rate. At a nominal 2.1% arithmetic annual return, the Canadian bond market forecast needs both a low ERP and a continuing low proxy for the risk free rate to justify
a 55% equity allocation. Otherwise, with a more reasonable ERP of 5%, for example, the equity allocation is 65%, while if the ERP is decoupled from the impact of unconventional monetary policy and instead an historic average equity return of 11.37% is used the equity allocation is 76%. Further, as the proxy for the risk free rate increases the result is that the bond market’s Sharpe ratio decreases faster than that for the equity market. This in turn pushes the equity market allocation closer to 100%.

This type of sensitivity analysis of forecast data to standard equity allocations is easy to do with the efficient set algorithm. This is not to say it is perfect, since the basic problem is one of incorporating both the proxy for the risk free rate and the return on the bond market within a single period model.
References


Roy, A. D. (1952), Safety first and the holding of assets, *Econometrica*, 20-3,


Table 1

Canadian Returns 1934-2016

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<th>CPI</th>
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Table 2

Optimal Portfolios

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<tr>
<td>Equity Sharpe ratio</td>
<td>0.414</td>
</tr>
<tr>
<td>Equity allocation</td>
<td>50.5%</td>
</tr>
<tr>
<td>&quot;Market&quot; expected return</td>
<td>0.090</td>
</tr>
<tr>
<td>&quot;market&quot; stdev</td>
<td>0.095</td>
</tr>
<tr>
<td>Market Sharpe ratio</td>
<td>0.471</td>
</tr>
<tr>
<td>MinVar Expected return</td>
<td>0.077</td>
</tr>
<tr>
<td>Minvar Stdev</td>
<td>0.080</td>
</tr>
<tr>
<td>MinVar Sharpe</td>
<td>0.396</td>
</tr>
<tr>
<td>Min Var weights</td>
<td>23.2%</td>
</tr>
<tr>
<td>&quot;60:40 Expected return</td>
<td>0.094</td>
</tr>
<tr>
<td>&quot;60:40 Sharpe ratio</td>
<td>0.466</td>
</tr>
</tbody>
</table>
Table 3

**Canadian Real Returns 1934-2016**

<table>
<thead>
<tr>
<th></th>
<th>T Bills</th>
<th>Equities</th>
<th>Bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>0.91</td>
<td>7.63</td>
<td>2.96</td>
</tr>
<tr>
<td>Median</td>
<td>1.56</td>
<td>8.28</td>
<td>3.16</td>
</tr>
<tr>
<td>Compound</td>
<td>0.84</td>
<td>6.31</td>
<td>2.53</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>3.69</td>
<td>16.43</td>
<td>9.55</td>
</tr>
<tr>
<td>Skew</td>
<td>-0.72</td>
<td>-0.31</td>
<td>0.46</td>
</tr>
<tr>
<td>Standard Error</td>
<td>0.40</td>
<td>1.80</td>
<td>1.05</td>
</tr>
</tbody>
</table>

Table 4

**AOH Hewitt January 2016 10 Year Forecast Returns**

<table>
<thead>
<tr>
<th></th>
<th>Average</th>
<th>Compound</th>
<th>Standard deviation</th>
<th>Sharpe ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inflation</td>
<td>1.90%</td>
<td>1.90%</td>
<td>1.30%</td>
<td></td>
</tr>
<tr>
<td>Treasury bills</td>
<td>1.90%</td>
<td>1.90%</td>
<td>1.00%</td>
<td></td>
</tr>
<tr>
<td>Long Term Canadas</td>
<td>2.10%</td>
<td>1.60%</td>
<td>9.90%</td>
<td>0.0202</td>
</tr>
<tr>
<td>Long Term bonds</td>
<td>2.50%</td>
<td>2.00%</td>
<td>10.80%</td>
<td>0.05556</td>
</tr>
<tr>
<td>Canadian equities</td>
<td>8.30%</td>
<td>7.10%</td>
<td>17.00%</td>
<td>0.37647</td>
</tr>
<tr>
<td>US equities</td>
<td>7.60%</td>
<td>6.50%</td>
<td>15.90%</td>
<td>0.35849</td>
</tr>
<tr>
<td>Global equities</td>
<td>7.80%</td>
<td>6.90%</td>
<td>14.70%</td>
<td>0.40136</td>
</tr>
</tbody>
</table>

Source: AON-Hewitt Capital Market Assumptions & Methodology (Canadian version)
### Table 5

**Optimal Portfolios**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bond Sharpe ratio</strong></td>
<td>0.020</td>
</tr>
<tr>
<td><strong>Equity Sharpe ratio</strong></td>
<td>0.376</td>
</tr>
<tr>
<td><strong>Equity allocation</strong></td>
<td>91.6%</td>
</tr>
<tr>
<td><strong>&quot;Market&quot; expected return</strong></td>
<td>0.078</td>
</tr>
<tr>
<td><strong>&quot;market&quot; stdev</strong></td>
<td>0.156</td>
</tr>
<tr>
<td><strong>Market Sharpe ratio</strong></td>
<td>0.377</td>
</tr>
<tr>
<td><strong>MinVar Expected return</strong></td>
<td>0.037</td>
</tr>
<tr>
<td><strong>Minvar Stdev</strong></td>
<td>0.086</td>
</tr>
<tr>
<td><strong>MinVar Sharpe</strong></td>
<td>0.207</td>
</tr>
<tr>
<td><strong>Min Var weights</strong></td>
<td>25.3%</td>
</tr>
<tr>
<td><strong>&quot;60:40 Expected return</strong></td>
<td>0.058</td>
</tr>
<tr>
<td><strong>&quot;60:40 Sharpe ratio</strong></td>
<td>0.358</td>
</tr>
</tbody>
</table>

### Table 6

**Optimal Equity Allocation**

<table>
<thead>
<tr>
<th>ERP %</th>
<th>Risk Free Rate</th>
<th>Exp Return</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0% 1% 2% 3%</td>
<td>MinVar</td>
</tr>
<tr>
<td>3</td>
<td>45.2% 55.8% 91.3% -379.2%</td>
<td>2.9%</td>
</tr>
<tr>
<td>4</td>
<td>49.6% 61.3% 93.3% 694.8%</td>
<td>3.1%</td>
</tr>
<tr>
<td>5</td>
<td>53.4% 65.3% 91.3% 283.5%</td>
<td>3.4%</td>
</tr>
<tr>
<td>6</td>
<td>56.7% 68.6% 95.3% 208.5%</td>
<td>3.6%</td>
</tr>
<tr>
<td>7</td>
<td>59.5% 71.4% 96.1% 177.0%</td>
<td>3.9%</td>
</tr>
<tr>
<td>8</td>
<td>62.0% 73.2% 96.4% 156.9%</td>
<td>4.1%</td>
</tr>
<tr>
<td>9</td>
<td>64.2% 75.7% 96.9% 148.7%</td>
<td>4.4%</td>
</tr>
</tbody>
</table>
Figure 1

Interest Rates and Inflation