Margin-restricted Mean-Variance Portfolio Deleveraging Under Market Impact*

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We consider the problem of deleveraging a large long-short portfolio of risky assets in a relatively short trading period under liquidity impact on prices. Given an adverse outlook on uncertainty in the one-period investment horizon, an optimally-deleveraged portfolio must be determined according to specified risk aversion and satisfying stricter leverage and margin policy constraints. Liquidity costs are considered due to both volume and trading intensity, leading to temporary and permanent impact on asset prices. The resulting portfolio model generalizes the usual mean-variance model; however, it has no closed-form solution nor is it possible to solve numerically with standard methods. One main contribution of the paper is to show how an efficient solution methodology is developed to obtain an optimally-deleveraged portfolio. The second main contribution is to employ the methodology to perform an empirical evaluation of the deleveraging model to develop insights on setting leverage and margin policy in deteriorating markets in which large and leveraged portfolios become highly-vulnerable, such as those witnessed during recent financial crises. We demonstrate our approach using a portfolio of ETF assets, and analyze the sensitivity of the optimal deleveraging strategy to leverage and margin limits under scenarios of market illiquidity. The study of MV efficient frontiers under liquidity costs allows us to develop managerial insight for setting deleveraging policy parameters. Empirical quantification of the relative gain from our approach, in comparison to ignoring market liquidity, under out-of-sample analysis is considered in our continuing research.

**Key words:** Portfolio deleveraging, market liquidity and trading impact, nonconvex quadratic optimization, M-V efficient frontiers.

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1. **Introduction**

As evidenced by historical market crashes, leveraged and margin-enabled portfolios of high NAV encounter increased pressure to reduce portfolio risk exposure in adverse market situations. For example, in the crash of 2009, Lehman Brothers portfolio with a leverage ratio of over 31:1 failed to raise sufficient capital to sustain the portfolio losses brought on by the rapidly falling asset

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prices under extreme pressures of illiquidity, leading to the company’s demise. Likewise, during the financial crisis of 1998, Long-Term Capital Management which had once leveraged their equity as much as 30:1 lost a staggering 44% of its equity during the month of August due to massive losses under forced liquidation of the holdings to cover margin requirements (Jorion 2000). The asset management industry is fraught with many such historical events associating financial ruin and high leverage during market turbulence. Hence, prudent portfolio deleveraging and management practices are indispensable under impending market downturns.

Using leverage (i.e. borrowing on margin for asset purchase) is often a mechanism used to realize the full growth potential of an investment strategy. While the preceding historical artifacts have led to the connotation that leverage is dangerous, leverage in itself is not the main cause for alarm, but it is the underlying portfolio’s (volatility) risk which indeed is amplified under leverage. That is, leveraging a high-risk unlevered portfolio carries much danger relative to a low-risk portfolio being leveraged to the same degree to improve target return. In addition, sufficient collateral should be available through long asset/cash positions to support portfolio short positions and avoid forced liquidation.

As market conditions evolve, a well-structured portfolio at the present time may become poorly-structured with regard to the above concerns in future adverse market scenarios, accompanied with elevated volatilities. Then, it is imperative that the portfolio is deleveraged to satisfy the borrowing policy and restructured for an acceptable risk-return profile whilst achieving a revised long-short balance under the specified margin restrictions. For funds with significant NAV, such a rebalancing operation also encounters elevated asset trading illiquidities during market downturn scenarios, thus, exacerbating the negative portfolio effects. As such, the portfolio deleveraging and rebalancing process must consider risk-return trade-off, leverage and margin limits, along with liquidity impact in trading ex-ante when determining optimal asset positions. This is the premise of the model and analysis presented in this paper.

Under Modern Portfolio Theory (MPT), see Markowitz (1952), MV optimal portfolios may be endowed with long-short leverage levels inconsistent with the investor’s leverage aversion, especially at higher portfolio volatility levels. In extending the MPT, Jacobs and Levy (2012, 2013, 2014) proposed to augment the mean-variance utility with a leverage risk averse component so that optimal portfolios so-determined are representative of the investor’s leverage preference. This is tantamount to indirect control of long-short exposures to satisfy margin requirements. As Asness et al. (2012) points out portfolios of safer assets combined in low levels of long-short leveraging may be further invested under external borrowing to improve returns, the latter referred to as

\[1\] Within the investment industry, the hedge fund sector makes the most use and employs high levels of leverage (Ang et al. 2011).
financial leveraging. This indeed is the rationale behind the so-called risk parity portfolios which allocate portfolio risk equally among different asset classes, and under financial-leveraging those less risky assets are further invested upon, i.e., overweighting safer assets. However, we note that these discussions have not incorporated liquidity impact, and that even the risk parity portfolios may end up overly-leveraged and in violation of margin policies when markets evolve under negative scenarios. The work in our paper is applicable for deleveraging such portfolios to rid them of excessive borrowing in the risky and riskfree assets in those markets punctuated with substantial illiquidity within the framework of MV utility.

Our work may also be viewed as a generalization of the model by Brown et al. (2010) who developed a deleveraging model that maximizes net equity of a long-only portfolio while satisfying a target leverage ratio. Although their model incorporates trading impact via temporary and permanent impact on asset prices based on the quantity and intensity of trading, it is free of long-short leveraging or any portfolio risk. Price impact is only based on asset sales for deleveraging long portfolios. Moreover, they assume a condition on the price impact parameters so that the resulting optimization model is guaranteed to be a convex, separable quadratic problem with a single constraint, solvable with relative ease. Chen et al. (2014) relaxes the latter assumption and develop an algorithm to solve the resulting indefinite separable singly-constrained quadratic program. Chen et al. (2015) extend the model for the case when temporary price impact is a higher-order power function of the trading rate.

The deleveraging models in the foregoing studies assume that the initial portfolio holds only long positions, and thus, deleveraging is to sell of assets to satisfy a financial leverage constraint. Our work relaxes this assumption and allows portfolio long and short positions; this implies that deleveraging involves not only selling, but also cover-buying. Our deleveraging strategy is limited to partial sale in long positions and purchases in short positions (partial cover buying). That is, in the deleveraged portfolio, neither the initial long positions become short nor initial short positions become long. Moreover, collateral using marginable long asset positions must be available to cover some or all of the credit risk that the short portfolio poses for the counterparty. The allowable margin could be a different percentage for each asset, and in order to avoid margin call if the balance available falls below the amount allowed, the short portfolio is subject to margin limits (Brumm et al. 2015). Consequently, the problem in this paper becomes a multi-constrained, nonconvex, mean-risk optimization model.

The second important contribution is that we consider the deleveraging problem in a utility framework. In the earlier work mentioned above, deleveraging (within a short time window) is performed to maximize the total asset value at the end of the deleveraging period without considerations of portfolio risk resulting from market evolution in the ensuing period. For instance,
a portfolio deleveraged over a day would still be held over a longer period (e.g., several weeks), resulting in unacceptable portfolio risk due to improperly-chosen assets and positions so-reduced. Portfolios deleveraged without any account of risk is contrary to the safe-betting concept in risk parity. Consequently, in our deleveraging model both long and short asset positions are shrunk simultaneously under liquidity costs to meet specified leverage and margin limits whilst achieving an efficient deleveraged portfolio in a mean-risk framework considering the portfolio performance outside the deleveraging period.

Also, significantly, we focus on developing guidance on setting leverage/margin policy parameters judiciously as market uncertainties evolve so that financial turmoil for the institution or the fund can be avoided. This requires developing insights on the trade-off frontier that results from policy parameters and portfolio risk-return characteristics. This necessitates solving the proposed deleveraging optimization model repeatedly given that the model is indefinite, nonseparable quadratic, and multi-constrained. Solving optimization models with an indefinite objective and multiple indefinite constraints even once is notoriously-difficult, especially when the number of variables (i.e. assets) is large, and thus, we develop a novel and efficient solution methodology, another significant contribution in this paper.

The solution methodology we develop in this paper is based on two steps. First, we develop a general theory for a Lagrangian dual cutting plane (DCP) technique that is computationally-efficient when the number of constraints is sufficiently small as in this application. The DCP method enables efficient solution of the Lagrangian dual deleveraging problem. Second, when a nonzero duality gap exists, using the Lagrangian optimal solution, we develop the theory to determine successively-improving feasible portfolio solutions, computable via solving a sequence of convex optimization problems. These convex separable programs too can be globally-solved efficiently using the DCP technique. Consequently, our solution method allows efficient numerical evaluation of the complete trade-off frontier to develop managerial insights on setting policy parameters.

Applying the solution methodology, we test the optimal deleveraging model empirically using a portfolio constructed with ETF assets. While ETFs used here are more liquid assets, our results provide compelling evidence of the role leverage and margin policy parameters play under trading impact when down-sizing a portfolio with a large NAV in countering the effects of an impending market downturn. We show how the MV efficient frontiers are affected under deleveraging to stricter policies, and the serious over-estimation that occurs when liquidity impact is ignored. We test scenarios of increased liquidity costs and limited collateral availability to ascertain the value of prudent deleveraging policy, and provide insights for asset managers.
2. Model Development

Consider a one-period economy from date 0 to \( T \) with a given portfolio \( x_0 \in \mathbb{R}^n \) of \( n \) risky assets at date 0, where the initial asset (share) positions are denoted by \( x_{0j}, j = 1, \ldots, n \). At the initial asset price vector \( p_0 \in \mathbb{R}^n \), the leverage level and margin requirements of portfolio \( x_0 \) may violate the institution policy. Regardless, we assume a negative asset/market return regime during \((0, T]\) compels the portfolio \( x_0 \) to be deleveraged by reducing position sizes, with trade executions during the time period \((0, 1]\) to positions (decision) vector \( x_1 \in \mathbb{R}^n \) at date 1, e.g., a monthly-portfolio with one-day deleveraging trade execution. While the decision \( x_1 \) is made at date 0, the execution of positions \( x_0 \rightarrow x_1 \) faces liquidity risks depending on the (continuous) trading trajectory \( t \rightarrow x_{tj}, t \in (0,1] \), corresponding to the trading rate \( y_{tj} \) where \( x_{1j} - x_{0j} = \int_0^1 y_{tj} dt \). If the trade-size \( ||x_1 - x_0|| \) is large (as is the case with high NAV), portfolio trading may face significant permanent price impact due to market liquidity; if the position transition occurs at a high intensity, denoted by the instantaneous rate of trading \( y_{tj} = \frac{dx_{tj}}{dt} \), then trading may also face temporary liquidity shortages, adversely affecting the trading price.

We assume \( T \gg 1 \), and thus, the market uncertainty in the execution period \((0,1]\) is negligible relative to uncertainties of the period \([1, T]\). As such, the price of asset \( j \) changes deterministically to \( p_{tj} \) at time \( t \in (0,1] \), for \( j = 1, \ldots, n \), due to the portfolio trading action. However, portfolio performance is evaluated at the terminal date \( T \) captures uncertainties outside the deleveraging period based on price uncertainty using risk-averse utility. We shall assume that asset returns during \([1, T]\) to be normally-distributed with \( r \sim \mathcal{N}(\mu, V) \), where \( \mu = \mathbb{E}[r] \in \mathbb{R}^n \) is the mean vector and \( V = \text{Var}[r] \in \mathbb{R}^{n \times n} \) is the covariance matrix.

As indicated, the price change of an asset during deleveraging has a permanent impact component that depends on the cumulative amount traded up until \( t \); on the other hand, its temporary impact component depends on the rate at which the asset is traded and it is instantaneous and reversible. This single asset price evolutionary model in Carlin et al. (2007) was extended to a portfolio of multiple assets by Brown et al. (2010) - also see Chen et al. (2014):

\[
\begin{align*}
    p_t &= p_0 + \Gamma(x_t - x_0) + \Lambda y_t, \quad t \in (0,1) \\
    p_1 &= p_0 + \Gamma(x_1 - x_0),
\end{align*}
\]

(1)

where the diagonal matrices \( \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n) \) and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \), and \( \gamma_j \) and \( \lambda_j \) denote the (positive) permanent impact and temporary impact coefficients, respectively. The asset pricing model (1) assumes that during the short trading period there is no growth in the asset price, and the price changes are effected only by the investor’s trading action. Extensions with stochastic price variation during the trading period have been proposed.\(^2\)

\(^2\)Almgren and Chriss (2000), Almgren (2003) split a single asset price into a market impact component due to trading and an unaffected price component that evolves according to the discrete arithmetic random walk. Gatheral and Schied (2012) employed a geometric Brownian motion (GBM) for the unaffected asset prices.
Trading rate in an asset is positive, i.e., purchase, only if the asset has a short position at time 0; \( y_{tj} \leq 0 \) indicates asset sales for an initial long position, i.e., \( x_{0j} > 0 \). Stated formally:

**Assumption 1.** Given an asset \( j = 1, \ldots, n \), \( y_{tj} \geq 0 \) for \( t \in [0, 1) \) and \( x_{1j} \in [x_{0j}, 0] \) if \( x_{0j} < 0 \). Moreover, \( y_{tj} \leq 0 \) for \( t \in [0, 1) \) and \( x_{1j} \in [0, x_{0j}] \) if \( x_{0j} > 0 \). If \( x_{0j} = 0 \), then \( y_{tj} = 0 \) for \( t \in [0, 1) \) and \( x_{1j} = 0 \).

A dynamic strategy is one in which the trade size depends on the stock price during execution of the order, such as in the case of a Delta-hedging strategy. For the special case of a single stock liquidation problem, Almgren and Chriss (2000) showed that under arithmetic Brownian motion, a static strategy is optimal - one that is determined in advance of trading, e.g. a constant trading rate, which is a volume-weighted average price (VWAP) approach. If the price process has no random term or the random component is independent of the current stock price, then a statically-optimal strategy will be dynamically-optimal for the problem of a single asset pure liquidation problem (Gatheral and Schied 2012). Motivated by this, since the price process during the execution period has no random term in our case, we follow Brown et al. (2010) and Chen et al. (2014) and employ the static constant-rate trading strategy, i.e., \( y_j \equiv y_{tj} = \frac{x_{1j} - x_{0j}}{1} = x_{1j} - x_{0j} \). Then, the net cash ‘generated’ during the (short) deleveraging period is:

\[
K(x_1) = -\int_0^1 p_1^\top y \, dt = -\int_0^1 [p_0^\top y + y^\top \Lambda y + ty^\top \Gamma y] \, dt = -x_1^\top M x_1 + 2(M x_0)^\top x_1 - x_0^\top M x_0 - p_0^\top (x_1 - x_0),
\]

where \( M := \Lambda + 0.5 \Gamma \) is a positive definite matrix since \( \Gamma \) and \( \Lambda \) are positive definite, i.e., \( \lambda_j, \gamma_j > 0 \).

Denote the initial (cash) liability at day 0 of the portfolio by \( L_0 \). A positive or negative \( L_0 \) indicates an initial debt level or surplus cash position, respectively. Assuming a positive initial portfolio net wealth at date 0,

\[
w_0 := p_0^\top x_0 - L_0 \quad (> 0).
\]

### 2.1. Leverage control

In the case of long-short portfolios, deleveraging may be ‘forced’ on the portfolio due to market conditions that threaten portfolio equity because such equity is used as collateral in creating short positions. To avoid forced liquidation (due to margin calls), additional capital may be borrowed exogenously, increasing portfolio liability. Portfolio managers resort to riskfree borrowing (hence, increasing debt to equity) to increase their exposure to risky assets in an attempt to increase target expected returns. In particular, modern Risk Parity strategies exploit this ‘financial leverage’ opportunity under exogenous borrowing to improve portfolio risk-adjusted performance. Using the
portfolio’s debt to equity (D/E) ratio as a mechanism to guide portfolio performance is hereby referred to simply as leverage control.

On the other hand, degree of infusion of additional capital to increase collateral also depends on the specific assets in the portfolio since more-risky assets may contribute differently to collateral requirement than less-risky assets. Therefore, if the portfolio is of long-short type, then the short portfolio must have adequate collateral support from the long portfolio. The extent of short portfolio value relative to the long portfolio value must be managed for controlling margin requirements (or ‘portfolio leverage’). We shall consider both of these leverage control mechanisms as they provide different (but related) portfolio protections.

For example, for the initial holdings at date 0, D/E (financial) leverage ratio $\rho_0$ is:

$$\rho_0 := \frac{p_0^\top x_0 + L_0^+}{p_0^\top x_0 - L_0^-},$$

(4)

where we use the following notation: $a^+ := \max\{a, 0\}$, $a^- = \max\{-a, 0\}$, and vector $b^-$ has its each component as $b_j^-$. Total portfolio debt comprising the cash liability and the short portfolio are considered in (4). Even if $\rho_0$ is an presently acceptable, portfolio deleveraging may still be required in anticipation of future market movement. In doing so, portfolio D/E leverage at date 1 must be brought under a prescribed level. For this, let:

$$L_1(x_1) = p_1^\top x_1 + [L_0 - K(x_1)]^+$$

$$= p_1^\top x_1 + [x_1^\top M x_1 + (p_0 - 2Mx_0)^\top x_1 - \phi]^+,$$

(5)

where

$$\phi := w_0 - x_0^\top M x_0$$

(6)

and the initial wealth $w_0$ is given by (3). Observe that $\phi$ is the portfolio net wealth if the initial positions $x_0$ are completely liquidated, since (2) yields the cash generated under liquidation as $K(0) = -x_0^\top M x_0 + p_0^\top x_0$, which when combined with the initial liability, we have $K(0) - L_0 = \phi$. In the sequel, we shall assume that the fully-liquidated portfolio has nonnegative wealth, i.e., $\phi \geq 0$.

The period-ending net equity (asset) position at date 1 is

$$A_1(x_1) = p_1^\top x_1 - [L_0 - K(x_1)]$$

$$= x_1^\top (\Gamma - M)x_1 + [(2M - \Gamma)x_0]^\top x_1 + \phi.$$

(7)

Thus, to satisfy the leverage ratio within a prescribed threshold $\rho$, the following constraint is imposed on portfolio rebalancing:

$$\frac{L_1(x_1)}{A_1(x_1)} \leq \rho.$$

(8)

Jacobs and Levy (2012) define portfolio leverage for long-short positions using the sum of absolute portfolio weights minus one, which is then incorporated into the objective utility function; also, see Jacobs and Levy (2013).
For the initial portfolio \( x_0 \), denote the index set of long assets by \( P \) and that of short assets by \( N \). Then, the above constraint can be expressed as the following two constraints:

\[
\begin{align*}
\rho A_1(x_1) + \sum_{j \in N} p_{1j} x_{1j} & \geq 0 \\
\rho A_2(x_1) + \sum_{j \in N} p_{1j} x_{1j} & \geq -\sum_{j \in P} \gamma_j + (p_0 - 2Mx_0)^\top x_1 - \phi,
\end{align*}
\]

where \( p_1 = p_0 + \Gamma(x_1 - x_0) \).

Remark: Noting (2), \( K(x_1) \) achieves a maximum (cash value), denoted by \( K_{\text{max}} \) when all long positions are liquidated, while all short positions remain unchanged, i.e., \( x_{1j} = 0, \forall j \in P \), and \( x_{1j} = x_{0j}, \forall j \in N \). Similarly, \( K(x_1) \) achieves a minimum cash position, denoted by \( K_{\text{min}} \), when all short positions are covered and all long positions remain unchanged, i.e., \( x_{1j} = 0, \forall j \in N \), and \( x_{1j} = x_{0j}, \forall j \in P \). That is,

\[
K_{\text{max}} = \sum_{j \in N} \left[ -M_{jj}(x_0j)^2 + (2M_{jj}x_{0j} - p_{0j})x_{0j} \right] - x_0^\top Mx_0 + p_0^\top x_0, \tag{10}
\]

and

\[
K_{\text{min}} = \sum_{j \in P} \left[ -M_{jj}(x_0j)^2 + (2M_{jj}x_{0j} - p_{0j})x_{0j} \right] - x_0^\top Mx_0 + p_0^\top x_0 \tag{11}
\]

which yields \( \phi = K_{\text{max}} + K_{\text{min}} - L_0 \), see (6). Therefore, unless \( L_0 \in (K_{\text{min}}, K_{\text{max}}) \), one of the constraints in (9) can be removed. More specifically,

1. If \( K_{\text{max}} \leq L_0 \), then \( x_1^\top Mx_1 + (p_0 - 2Mx_0)^\top x_1 - \phi \geq L_0 - K_{\text{max}} \geq 0 \), and thus, the first constraint in (9) can be eliminated.

2. If \( K_{\text{min}} \geq L_0 \), then \( x_1^\top Mx_1 + (p_0 - 2Mx_0)^\top x_1 - \phi \leq L_0 - K_{\text{min}} \leq 0 \), and thus, the second constraint in (9) can be eliminated.

2.2. Margin restrictions

Since short positions in a portfolio can lose more than its initial value (unlike long positions), short-sales must be covered by sufficient margin collateral to control ‘excessive shortsale risk’, see Edirisinghe (2007). That is, the total short position must be controlled not to exceed a certain fraction of the total (marginable) long positions.

Suppose an asset \( j \) contributes up to a maximum of \( \eta_j \) margin collateral per dollar invested in a long position in asset \( j \), where \( \eta_j \in [0, 1], \forall j \). Upon portfolio deleveraging, the rebalanced portfolio has a total marginable collateral of \( \sum_{j \in P} \eta_j p_{1j} x_{1j} \), and the short portfolio value is \( p_1^\top x_1 = \)
\[-\sum_{j \in N} P_{1j}x_{1j},\] given in absolute dollar terms. A portfolio margin control policy at level $\zeta_1$ implies the constraint:

\[
- \sum_{j \in N} P_{1j}x_{1j} \leq \zeta_1. \tag{12}
\]

Observe that (12) is a margin policy that covers the entire group of assets under the fund’s management. On the other hand, the fund’s margin policy may also include additional protection against excessive short-sale risk in pre-specified asset groups, e.g., technology, finance, utilities, etc. That is, portfolio allocations within a given asset group must allow for ‘sufficient’ collateral to support short exposure in the group using long exposure within the group, see Edirisinghe (2007). For instance, in order to limit the excessive short-selling risk within a volatile sector, such as the Internet stocks, the managerial policy may be that no more than 30% of the long position in Internet stocks may be tied to short positions within the same sector.

Therefore, we generalize the margin constraint in (12) by considering $S$ asset groups, indexed by $s = 1, \ldots, S$, with $s = 1$ denoting the entire asset set, $P \cup N$. For $s = 2, \ldots, S$, suppose the initial portfolio $x_0$ has long and short positions in asset subsets, denoted by $P_s \subset P$ and $N_s \subset N$, respectively, and we have $P_1 = P$ and $N_1 = N$. This leads to a set of $S$ margin constraints that subsumes (12), for a specified margin policy parameter (nonnegative) vector $\zeta \in \mathbb{R}^S$:

\[
\sum_{j \in N_s} p_{1j}x_{1j} + \zeta_s \sum_{j \in P_s} \eta_j p_{1j}x_{1j} \geq 0, \quad s = 1, \ldots, S. \tag{13}
\]

Note that (13) is a set of $S$ number of nonconvex quadratic constraints since $p_1 = p_0 + \Gamma(x_1 - x_0)$ and $P_s \neq \emptyset$.

2.3. Expected utility maximization

Subject to satisfying the leverage and margin constraints, the portfolio deleveraging is performed to maximize the expected utility of the net asset value at date $T$, denoted by the random variable, $A_T(x_1)$. We shall assume that the uncertainty evolves independent of the investor’s deleveraging action, over the period $[0, T]$, denoted by a price change of $\Delta_j$ per share of asset $j$. That is, $\Delta_j = r_jp_{0j}$, where $r_j$ is the asset’s random return over $[0, T]$. We approximate the random asset return over the period $[1, T]$ by $r_j$ since $T \gg 1$. Accordingly, the random dollar return on the risky portfolio in $[1, T]$ is $\sum_{j=1}^n \Delta_j x_{1j}$, while the risk-free component of the portfolio yields $e^{r_0} [K(x_1) - L_0]$, where we assume that cash can be borrowed or lent at the continuously-compounded risk-free rate, $r_0(\geq 0)$, per period $T$.

Defining the diagonal matrix $P_0 = \text{diag}(p_{01}, \ldots, p_{0n})$, thus,

\[
A_T(x_1) = A_1(x_1) + P_0 r^T x_1 + (e^{r_0} - 1) [K(x_1) - L_0]. \tag{14}
\]
Define the portfolio return by $R_T(x_1) := \frac{1}{w_0} A_T(x_1)$. Under normally-distributed $r$ and CRRA utility function $U$ with risk aversion parameter $\vartheta > 0$, maximizing $\mathbb{E}[U(A_T(x_1))]$ is equivalent to determining the optimal trade-off:

$$\max \mathbb{E}[R_T(x_1)] - \vartheta \text{Var}[R_T(x_1)], \quad (15)$$

subject to the leverage and short-sale margin constraints in (9) and (14), respectively, satisfied under the policy-pair $(\rho, \zeta)$. Noting that

$$\text{Var}[R_T(x_1)] = \frac{1}{(w_0)^2} x_1^\top P_0 V P_0 x_1, \quad (16)$$

and the portfolio expected return at date $T$:

$$\mathbb{E}[R_T(x_1)] = \frac{1}{w_0} \left[ A_1(x_1) + P_0 \mu ^\top x_1 + (e^0 - 1) [K(x_1) - L_0] \right]$$

$$= \frac{1}{w_0} \left[ x_1^\top (\Gamma - M) x_1 + [P_0 \mu + (2M - \Gamma) x_0] ^\top x_1 + (e^0 - 1) [K(x_1) - L_0] + \phi \right]$$

$$= \frac{1}{w_0} \left[ x_1^\top (\Gamma - e^0 M) x_1 + [P_0 \mu + (2e^0 M - \Gamma) x_0 - (e^0 - 1)p_0] ^\top x_1 + e^0 \phi \right]. \quad (17)$$

the ‘market impact deleverage optimization’ (MIDO) model is given by:

$$F(\rho, \zeta, \vartheta) := \max_{x_1} \frac{1}{w_0} \left[ x_1^\top (\Gamma - e^0 M) x_1 + [P_0 \mu + (2e^0 M - \Gamma) x_0 - (e^0 - 1)p_0] ^\top x_1 \right] - \frac{\vartheta}{(w_0)^2} x_1^\top P_0 V P_0 x_1$$

s.t. $\rho A_1(x_1) + \sum_{j \in N} p_{1j} x_{1j} \geq 0$

$\rho A_1(x_1) + \sum_{j \in N} p_{1j} x_{1j} - x_1^\top M x_1 - (p_0 - 2M x_0) ^\top x_1 + \phi \geq 0 \quad (18)$

$$\sum_{j \in N_s} p_{1j} x_{1j} + \zeta_s \sum_{j \in P_s} \eta_{1j} p_{1j} x_{1j} \geq 0, \ s = 1, \ldots, S$$

$$0 \leq x_{1j} \leq x_{0j}, \ j \in P; \ x_{0j} \leq x_{1j} \leq 0, \ j \in N.$$

We observe that (18) is a computationally-tedious optimization model since the constraints have quadratic non-convexities and the objective may also be quadratic non-convex. While the portfolio variance term is convex, it is nonseparable in the positions $x_1$. In the ensuing sections, we develop an efficient solution method for (18). Feasibility of MIDO can be assured under mild conditions.

**Proposition 1.** For the initial portfolio $(x_0, L_0)$, assume that the fully-liquidated portfolio has non-negative wealth, i.e., $\phi \geq 0$, see (6). Then, the feasible set of MIDO model (18) is nonempty. In particular, the fully-liquidated portfolio $x_1 = 0$ is feasible in (18).

Proof is omitted because it is straightforward. Let an optimal solution of (18) be denoted by $x_1^*$.

We are particularly interested in the trade-off frontier between $E[A_T(x_1^*)]$, see (14), and portfolio standard deviation $\sigma_p(x_1^*) := \sqrt{x_1^*^\top P_0 V P_0 x_1^*}$, for fixed policy $(\rho, \zeta)$ as $\vartheta$ is varied, as well as the sensitivity of the frontier on the latter policy, using an implementation with real-world data in Section 3. The next two sections are devoted to developing algorithms to compute upper and lower bounds on $F(\rho, \zeta, \vartheta)$. 

2.4. Model without market impact, MVO

The mean-variance optimization (MVO) model disregarding market impact, but subject to leverage and margin constraints, is also considered to evaluate the degree of suboptimality and underperformance of portfolios so-determined by ignoring liquidity costs. The MVO model is:

\[
g(\rho, \zeta, \vartheta) := \max_{x_1} \frac{1}{w_0} \left[\mathbf{P}_0 \mu - (e^{r_0} - 1)p_0\right]^\top x_1 - \frac{\vartheta}{(w_0)^2} x_1^\top \mathbf{P}_0 \mathbf{V} \mathbf{P}_0 x_1
\]

s.t. \(\rho w_0 + \sum_{j \in N} p_{0j} x_{1j} \geq 0\)

\((1 + \rho)w_0 - \sum_{j \in P} p_{0j} x_{1j} \geq 0\)

\(\sum_{j \in N_s} p_{0j} x_{1j} + \zeta s \sum_{j \in P_s} \eta_j p_{0j} x_{1j} \geq 0, \ s = 1, \ldots, S\)

\(0 \leq x_{1j} \leq x_{0j}, \ j \in P; \quad x_{0j} \leq x_{1j} \leq 0, \ j \in N.\) (19)

Since (19) is a pure convex (quadratic) program with linear constraints, it can be directly solved using CPLEX V12.7.1. Suppose an optimal solution of the above convex model is \(\hat{x}_1.\) Indeed, \(g(\rho, \zeta, \vartheta) \geq F(\rho, \zeta, \vartheta)\) must hold. However, \(\hat{x}_1\) may become infeasible in the MIDO model (18) that incorporates market impact costs in the constraints, and thus, \(\hat{x}_1\) may not be implementable. In the event \(\hat{x}_1\) is feasible in MIDO model, its objective value, denoted by

\[
\hat{g}(\hat{x}_1) \equiv \frac{1}{w_0} \left[\hat{x}_1^\top (\Gamma - e^{r_0} M) \hat{x}_1 + [\mathbf{P}_0 \mu + (2 e^{r_0} M - \Gamma)x_0 - (e^{r_0} - 1)p_0]^\top \hat{x}_1\right] - \frac{\vartheta}{(w_0)^2} \hat{x}_1^\top \mathbf{P}_0 \mathbf{V} \mathbf{P}_0 \hat{x}_1,
\]

is inferior, i.e., \(\hat{g}(\hat{x}_1) \leq F(\rho, \zeta, \vartheta).\) In such a case, the solution \(\hat{x}_1\) may be used to initialize the lower bounding procedure in Section 2.5 to obtain an improved deleveraged solution, say \(\hat{x}_1^*\).

2.5. Bounds-based method to solve MIDO

Since \(p_1 = p_0 + \Gamma(x_1 - x_0),\) observe that the MIDO model (18) has \(S + 2\) quadratic separable constraints, in addition to the lower and upper limits on variables. Note that the \(S\) margin constraints are nonconvex, and also depending on the liquidity impact coefficient-values, leverage constraint and the objective function can become indefinite (or nonconvex). Thus, MIDO is a difficult quadratically-constrained indefinite quadratic programming (QCQP) model, see e.g. Bao et al. (2011), Linderoth (2005) for comprehensive reviews and surveys of general QCQP model solution algorithms. More recently, Lu et al. (2018) develop an algorithm using semi-definite relaxation based branch-and-bound method. These methods are not appropriate for funds with large number assets since these methods are largely ineffective for large \(n.\)

Instead, we propose a new dual cutting plane (DCP) technique for solving a Lagrangian dual that is quite efficient for the problem at hand given its almost-separable structure.\(^4\) The necessary theory
is developed in Section EC.1. The DCP algorithm computes the Lagrangian dual optimum value, which is an upper bound on $F(\rho, \zeta, \vartheta)$ due to duality gap. After determining the dual optimum portfolio, we develop a method to generate lower bounding feasible portfolios for (18) that can converge to a near-optimal feasible deleveraging strategy of the MIDO model.

To compute the dual upper bounds using the DCP method, all constraints of (18) are Lagrangian-relaxed except for the bounds on variables. Using the stylized problem in (EC.24), for some $\theta^t \in \Theta$ at iteration $t$, we have the DCP subproblem corresponding to (EC.18) as:

$$L^*_U(\theta^t) = \max_{l \leq x \leq u} \left( (1 - \sum_{i=1}^{m} \theta^t_i) \left[ \sum_{j \in J_0^+} d_{0j}(x_j)^2 + \sum_{j \in J_0^-} d_{0j}(x_j)^2 + c_0^T x - h(x) \right] - \sum_{i=1}^{m} \theta^t_i \left[ \sum_{j \in J_i^+} d_{ij}(x_j)^2 + \sum_{j \in J_i^-} d_{ij}(x_j)^2 + c_i^T x - b_i \right] \right).$$

(21)

Note that (21) is a nonconvex nonseparable quadratic program with 'box' constraints, which is a class of problems that has received significant attention in the literature, see the survey in De Angelis et al. (1997). Specific algorithms for 'Box QP' problems based on difference of convex functions (DC) are in An and Tao (1998), Cambini and Sodini (2005), and for more recent work based on semidefinite programming (SDP) relaxations, see e.g. Burer and Vandenberghe (2008, 2009), Buchheim and Wiegele (2013). However, in our implementation, we use IBM\textsuperscript{TM} CPLEX V12.7.1, and obtain a global optimal solution $x^*_t$ of (21). Then, the cut $\sum_{i=1}^{m} \alpha^*_i \theta^t_i \leq \beta^t$ is imposed on $\Theta$ to obtain the reduced search space $\Theta_t$, see (EC.22)-(EC.23). Setting $\theta^{t+1} = C(\Theta_t)$, the centroid of $\Theta_t$, (21) is re-solved, and the procedure is terminated when $||\theta^{t+1} - \theta^t|| < \varepsilon$, with solution $x^{U}\_t$ and upper bound $F_U(\rho, \zeta, \vartheta)$ in hand.

To compute the lower bound using the DCP method, the model (EC.27) is Lagrangian-relaxed in the same way as (21), which yields the DCP subproblem:

$$L^*_L(\theta^t) = \max_{l \leq x \leq u} \left( (1 - \sum_{i=1}^{m} \theta^t_i) \left[ \sum_{j \in J_0^-} d_{0j}(x_j)^2 + 2 \sum_{j \in J_0^+} (d_{0j} y_j^k) x_j - h(x) \right] - \sum_{i=1}^{m} \theta^t_i \left[ \sum_{j \in J_i^+} d_{ij}(x_j)^2 + 2 \sum_{j \in J_i^-} (d_{ij} y_j^k) x_j - b_i + \sum_{j \in J_i^-} d_{ij}(y_j^k)^2 \right] \right).$$

(22)

Note that the subproblem (22) has a pure concave objective with box constraints, which can be solved efficiently to global optimality, which gives an upper bound on (EC.27). We shall employ CPLEX V12.7.1 in our implementation rather than using specialized software for convex programs with only box constraints. The upper bounding objective values on (EC.27), obtained via the DCP method, converge to the global optimal value because the model (EC.27) is a convex program and
it has no Lagrangian duality gap. That is, the DCP-based solution method determines the optimal value \( f_L(y^k) \), along with its optimal solution \( y^{k+1} \). Then, \( f(y^k) \) is an improved lower bound on the deleveraging model, MIDO. The lower bound is further-improved by forming a new lower bounding model using \( y^{k+1} \), which is solved using yet another iterative scheme of the DCP method. The process is terminated when \( \frac{f(y^{k+1}) - f(y^k)}{1 + |f(y^k)|} < \varepsilon \), say at some iteration \( K \), and we denote the lower bounding feasible solution by \( x_L^1 \equiv y^K \). Therefore, the (best) feasible deleveraged portfolio \( y^K \) is associated with an optimum-quality guarantee, w.r.t. \( F(\rho, \zeta, \vartheta) \) in (18), as measured by the relative error:

\[
\mathcal{E} := \frac{F_\theta(\rho, \zeta, \vartheta) - f(x_L^1)}{1 + |f(x_L^1)|}.
\]  

Note that in the process of determining \( \mathcal{E} \), the iterative Lagrangian-based DCP solution method has been used \( K + 1 \) times. In the application reported next, we compute the deleveraged portfolio \( x_L^1 \) with the optimality guarantee \( 1 - \mathcal{E} \).

3. Portfolio Deleveraging Application

The application of the model and methodology of the preceding sections employs data on nine sector-ETFs assets (ticker symbols XLB, XLE, XLF, XLI, XLK, XLP, XLU, XLV, and XLY), with trading period set to one day, and the trade holding period is set to 20 days, and thus, \( T = 21 \) days. The initial portfolio is deleveraged on Aug 03, 2015 and the rebalancing period end on Aug 31, 2015. These ETFs cover the full-breadth of the S&P 500 market index. An estimation procedure for the market impact parameters, using the millisecond TAQ data, is described in Edirisinghe et al. (2018) for the nine ETFs in Jan. 2015 - we shall use these estimates as a ‘base’ case of market impact which will be varied to analyze its effect on optimal portfolio deleveraging - see Table 1 for the base values of \( \gamma \) and \( \lambda \).\(^5\) In the same table, we report the forecasted mean and covariance parameters of asset and the market index retuns, for the 21-day period Aug 03-31, 2015, including those for the market index proxy ETF ticker, SPY, using historical data prior to Aug 03.

Initial portfolio (net risky) investment is \( p_0^T x_0 = $2m \), and the initial liability is \( L_0 = $1m \), hence the initial wealth is \( w_0 = p_0^T x_0 - L_0 = $1m \). The initial portfolio \( x_0 \) is also reported in Table 1 as portfolio dollar weights, computed by \( p_{0j}x_{0j}/p_0^T x_0 \), \( \forall j \). Thus, the long portfolio is \( p_0^T x_0^+ = $3m \), while the short portfolio value is \( p_0^T x_0^- = $1m \). This initial portfolio allocations reflect the market outlook predicted in the previous month, where the actual returns for the period Jul 01-31, 2015, are also reported in Table 1. Initial positions are largely long consistent with the current conditions, including the positive market index return (+1.5%) for the period Jul 01-31. However, a negative market performance (−0.3%) is predicted for Aug 03-31. Therefore, portfolio \( x_0 \) is going to be

\(^5\)The base values of \( \gamma \) and \( \lambda \) show that (\( \Gamma - M \)) is negative definite. Thus, under this case, the deleveraging model (18) is a concave maximization with three convex constraints and one nonconvex constraint.
deleveraged towards a new (optimal) portfolio $x_1$ during the trading day of Aug 03, and $x_1$ is held unchanged through Aug 31. Annualized risk-free rate is $r_0 = 2\%$ (used to compute the monthly-compounded rate).

### Table 1  Market impact parameters and asset return parameter forecasts for period Aug 03-31, 2015

<table>
<thead>
<tr>
<th>Parameter</th>
<th>XLB</th>
<th>XLE</th>
<th>XLF</th>
<th>XLI</th>
<th>XLK</th>
<th>XLP</th>
<th>XLU</th>
<th>XLV</th>
<th>XLY</th>
<th>SPY</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>base case: $\gamma \times 10^{-5}$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>base case</td>
<td>0.6361</td>
<td>0.7157</td>
<td>0.0127</td>
<td>0.2284</td>
<td>0.0294</td>
<td>0.1797</td>
<td>0.3204</td>
<td>0.5599</td>
<td></td>
<td>–</td>
</tr>
<tr>
<td><strong>base case: $\lambda \times 10^{-5}$</strong></td>
<td>4.3737</td>
<td>6.5823</td>
<td>3.776</td>
<td>2.8244</td>
<td>3.2523</td>
<td>4.2074</td>
<td>4.8528</td>
<td>4.6215</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean ($\mu$)</td>
<td>-0.038</td>
<td>-0.046</td>
<td>0.009</td>
<td>-0.019</td>
<td>-0.009</td>
<td>-0.007</td>
<td>0.013</td>
<td>0.015</td>
<td>-0.003</td>
<td></td>
</tr>
<tr>
<td>StDev ($\sigma$)</td>
<td>0.023</td>
<td>0.018</td>
<td>0.013</td>
<td>0.020</td>
<td>0.027</td>
<td>0.032</td>
<td>0.013</td>
<td>0.013</td>
<td>0.012</td>
<td></td>
</tr>
<tr>
<td>Asset beta ($\beta$)</td>
<td>0.155</td>
<td>0.174</td>
<td>0.907</td>
<td>0.859</td>
<td>1.458</td>
<td>1.358</td>
<td>1.376</td>
<td>0.878</td>
<td>0.969</td>
<td>1.000</td>
</tr>
<tr>
<td><strong>Asset correlations ($\rho$)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XLB (Basic Materials)</td>
<td>1</td>
<td>0.874</td>
<td>0.105</td>
<td>0.512</td>
<td>-0.120</td>
<td>-0.569</td>
<td>-0.607</td>
<td>0.461</td>
<td>-0.091</td>
<td>0.080</td>
</tr>
<tr>
<td>XLE (Energy)</td>
<td>1</td>
<td>0.040</td>
<td>0.509</td>
<td>-0.169</td>
<td>-0.363</td>
<td>-0.506</td>
<td>0.459</td>
<td>0.625</td>
<td>0.627</td>
<td>0.837</td>
</tr>
<tr>
<td>XLF (Financials)</td>
<td>1</td>
<td>0.699</td>
<td>0.839</td>
<td>0.290</td>
<td>0.207</td>
<td>0.625</td>
<td>0.837</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XLI (Industrial Goods)</td>
<td>1</td>
<td>0.660</td>
<td>0.196</td>
<td>0.143</td>
<td>0.848</td>
<td>0.67</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XLK (Technology)</td>
<td>1</td>
<td>0.473</td>
<td>0.498</td>
<td>0.586</td>
<td>0.647</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XLP (Consumer Staples)</td>
<td>1</td>
<td>0.883</td>
<td>0.303</td>
<td>0.807</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XLU (Utilities)</td>
<td>1</td>
<td>0.197</td>
<td>0.644</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XLV (Health Care)</td>
<td>1</td>
<td>0.679</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XLY (Consumer Discre.)</td>
<td>1</td>
<td>0.884</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Initial portfolio weights</strong></td>
<td>-12.5%</td>
<td>-12.5%</td>
<td>-12.5%</td>
<td>-12.5%</td>
<td>25.0%</td>
<td>37.5%</td>
<td>37.5%</td>
<td>12.5%</td>
<td>37.5%</td>
<td>0</td>
</tr>
<tr>
<td><strong>Open price $p_0$ (Aug 03)</strong></td>
<td>44.45</td>
<td>66.57</td>
<td>19.75</td>
<td>52.43</td>
<td>41.41</td>
<td>48.41</td>
<td>41.70</td>
<td>74.80</td>
<td>78.22</td>
<td>203.80</td>
</tr>
<tr>
<td><strong>Realized return (Jul 01-31)</strong></td>
<td>-0.056</td>
<td>-0.065</td>
<td>0.019</td>
<td>-0.002</td>
<td>0.022</td>
<td>0.044</td>
<td>0.055</td>
<td>0.019</td>
<td>0.037</td>
<td>0.015</td>
</tr>
</tbody>
</table>

Therefore, the initial (financial) leverage ratio is $\rho_0 = \frac{p_0^\top x_0^\top + L_0^\top}{w_0} = \frac{1+1}{1.0} = 2.0$, see (4). We will only consider a single margin constraint comprising the entire asset set, i.e., $S = 1$, and we denote $\zeta \equiv \zeta_1$. Assuming each long asset contributes 100% to margin, i.e., $\eta_i = 1$, the initial margin ratio for the entire asset set is, $\zeta_0 = \frac{\sum_{j \in N} p_{1j} x_{1j}^\top}{\sum_{j \in P} \eta_j p_{1j} x_{1j}} = 33.3\%$. If the portfolio is not deleveraged, due to the market downturn predicted through return parameters, the portfolio may worsen.\(^6\) Therefore, the effect of setting policy parameters, $\rho$ and $\zeta$ on leverage and margin, respectively, will be analyzed in determining an MV-optimal portfolio, $x_1^\ast(\rho, \zeta, \vartheta)$ for a given risk aversion $\vartheta > 0$, whose actual performance will be out-of-sample simulated during the (future) period Aug 04-31.

### 3.1. MIDO model analysis and efficient frontiers

Under the parameters in Table 1, the deleveraging model (18) for a given triple $(\rho, \zeta, \vartheta)$ is solved via the DCP method to determine the upper bound $F_\vartheta(\rho, \zeta, \vartheta)$ and the associated solution $x_1^\ast(\rho, \zeta, \vartheta)$. If the latter solution is feasible in (18), then it solves the MIDO model. Otherwise, starting with the infeasible portfolio $x_1^\ast(\rho, \zeta, \vartheta)$, a sequence of feasible portfolios is generated using the procedure...

\(^6\) If un-deleveraged, the initial portfolio’s net value on Aug 31, 2015 will be $0.887$m (long value=$2.833$m, short value=$0.944$m, cash liability=$1.002$m), a portfolio loss of 11.3% over the month. Furthermore, fund leverage and margin ratios will be 2.219 and 0.333 on Aug 31.
in Section EC.2, which converges to the most-improved feasible portfolio, \(x^*_L(\rho, \zeta, \vartheta)\), whose quality is given by \(1 - \mathcal{E}\), see (23).

Table 2 provides evidence of model solution quality \((1 - \mathcal{E})\), solution convergence, and portfolio characteristics for leverage/margin policy parameters set at two different parameter settings: reducing the (financial) leverage from current 2.0 to \(\rho = 1.0\), while changing margin ratio from current 33.3\% either by reducing to \(\zeta = 20\%\) or by increasing to \(\zeta = 60\%). We set the risk aversion parameter at \(\vartheta = 10\) for the sample model runs in Table 2, where ‘after trading’ terms are defined as follow: long (portfolio) value = \(p_1^T x_1^+\), short (portfolio) value = \(p_1^T x_1^-\), where \(p_1\) is given by (1), borrowed cash = \(L_0 - K(x_1)\), and trading loss = \(w_0 - p_1^T x_1 + L_0 - K(x_1)\). The initial portfolio in Table 1 has a standard deviation of 6.11\%.

In the case of applying the standard MV optimization (MVO) model ignoring trading impact, one sets \(\lambda = \gamma = 0\) model, and use (19) for the same lave of risk aversion, \(\vartheta = 10\). The MVO model is a pure convex quadratic optimization, which is solved using the standard software, IBM\textsuperscript{TM} CPLEX V12.7.1. Then, the optimal portfolio \(x_1\) so-obtained is used to calculate the cash generated \(K(x_1)\) by (2), and portfolio characteristics are computed based on what the price impact would have been under market impact, i.e., using \(p_1\) in (1). The base case of market impact is referred to as ‘high’ liquidity case.\(^7\)

As Table 2 shows ignoring liquidity impact (in this case, ‘high’ liquidity or \(1\times\) impact parameters) leads to undermining portfolio performance and increased trading costs, especially when deleveraging under stricter leverage policy. For example, with \(\rho = 1.0\) and \(\zeta = 0.6\), ignoring liquidity costs during portfolio deleveraging (using the MVO model) results in trading costs increasing by about 90\% over the MIDO model. The increased level of trading associated with the MVO-optimal portfolio is evident in the portfolio turnover metric, i.e., \(||x_1 - x_0||_2\), which is about 20\% more than that of the MIDO-optimal portfolio.

As market impact costs increase, such as the case when distressed markets encounter shortage of liquidity, the above-mentioned vulnerabilities of the MVO-optimal portfolio that do not account for illiquidity are magnified. To illustrate, the efficient frontiers of the monthly expected portfolio return and its standard deviation for the two models, MIDO and MVO (the ‘dotted’ curve), as well as the MVO-optimal portfolios when trading impact is incorporated ex-post, under ‘high’ or ‘low’ liquidity levels (1\(\times\) or 3\(\times\) impact parameters, respectively), are presented in Figure 1. Observe that ignoring market impact leads to an over-estimated MV frontiers; however, when trading costs are incorporated ex-post to account for executing the optimal MVO portfolio, its performance becomes significantly-inferior. On the other hand, the optimal MIDO portfolio incorporates liquidity costs

\(^7\)We will consider sensitivity on the level of market impact by considering 2\(\times\), 3\(\times\), and 4\(\times\) market impact parameters, referred to as ‘moderate’, ‘low’, and ‘very low’ liquidity cases.
Table 2  Illustration of deleveraged optimal portfolios on Aug 03, 2015 for ‘high’ liquidity and low risk aversion

<table>
<thead>
<tr>
<th>Model</th>
<th>Param.</th>
<th>Long</th>
<th>Short</th>
<th>Borrowed</th>
<th>Trading</th>
<th>Portfolio</th>
<th>Quality</th>
<th>UB iter</th>
<th>LB iter</th>
<th>E[A_T(x_t)]</th>
<th>σ_T</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(ρ, ζ)</td>
<td>(m$)</td>
<td>(m$)</td>
<td>(m$)</td>
<td>loss ($)</td>
<td>turnover</td>
<td>(1 − ζ)</td>
<td># DCP</td>
<td>(# DCP)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial</td>
<td>(2.0,0.33)</td>
<td>3.0</td>
<td>1.0</td>
<td>1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.887</td>
<td>6.11%</td>
</tr>
<tr>
<td>MIDO</td>
<td>(1.0,0.2)</td>
<td>1.826</td>
<td>0.183</td>
<td>0.654</td>
<td>11.19445</td>
<td>22,608</td>
<td>99.99%</td>
<td>77</td>
<td>3 (211)</td>
<td>1.014</td>
<td>3.38%</td>
</tr>
<tr>
<td>MIDO</td>
<td>(1.0,0.6)</td>
<td>1.863</td>
<td>0.559</td>
<td>0.313</td>
<td>9,075.40</td>
<td>21,482</td>
<td>99.99%</td>
<td>69</td>
<td>3 (204)</td>
<td>1.031</td>
<td>3.83%</td>
</tr>
<tr>
<td>MVO‡</td>
<td>(1.0,0.2)</td>
<td>1.477</td>
<td>0.148</td>
<td>0.349</td>
<td>19,830.03</td>
<td>26,797</td>
<td>100%</td>
<td>-</td>
<td>-</td>
<td>1.024</td>
<td>3.21%</td>
</tr>
<tr>
<td>MVO</td>
<td>(1.0,0.6)</td>
<td>1.648</td>
<td>0.494</td>
<td>0.171</td>
<td>17,259.66</td>
<td>25,626</td>
<td>100%</td>
<td>-</td>
<td>-</td>
<td>1.024</td>
<td>3.21%</td>
</tr>
</tbody>
</table>

Portfolio weights: XLB XLE XLF XLI XLK XLP XLU XLV XLY beta

| Initial | -12.50% | -12.50% | -12.50% | -12.50% | -12.50% | 25.00% | 37.50% | 37.50% | 12.50% | 37.50% | 1.60 |
| MIDO   | 0.00%   | -18.07% | 0.00%   | 0.00%   | 0.00%   | 56.57% | 27.16% | 25.26% | 75.59% | 2.06   |
| MIDO   | -25.03% | -24.70% | 0.00%   | -5.88%  | 0.00%   | 59.68% | 27.61% | 25.21% | 75.43% | 2.01   |
| MVO    | 0.00%   | -14.75% | 0.00%   | 0.00%   | 0.00%   | 48.70% | 0.00%  | 25.49% | 76.26% | 1.60   |
| MVO    | -24.66% | -24.90% | 0.00%   | 0.00%   | 0.00%   | 65.99% | 0.00%  | 25.42% | 76.06% | 1.77   |

1 Market impact parameters are set to the base case, and risk aversion ϑ = 10.
2 Portfolio turnover is measured by \[||x_1 - x_0||^2\] shares.
* Expected net wealth on Aug 31 under trading costs for optimal portfolios of MIDO or MVO models, see (17).
† MIDO model takes 3.006 cpu seconds on average.
‡ MV optimization (MVO) ignoring trading impact, i.e., λ = γ = 0, and it takes 0.003 cpu seconds on average.

Figure 1  Deleveraging mean-variance efficient frontiers (Target leverage, ρ = 1 and margin, ζ = 0.6)

Ex-ante and its performance stays superior at all levels of portfolio risk when market liquidity is ‘low’. At high levels of liquidity, the MVO-optimal portfolio’s performance is very close to that of the MIDO-optimal portfolio. This signifies the fact that when markets are not liquidity-distressed, investor’s opportunity costs of not considering trading impact ex-ante is quite negligible. In contrast, at low liquidity, there is a significant liquidity premium by considering trading impact ex-ante. For a situation with more strict margin control at ζ = 0.20, see Figure 2. Observe that in a market with high trading costs, along with stricter margin control, even the MIDO-optimal portfolio has difficulty in making a positive return in this (distressed) August month, while the MVO-optimal portfolio loses money even when the risk aversion is zero, i.e., the right-most point of the frontier.
Algorithmic performance of the upper and lower bounds developed in this paper, based on the DCP technique, appears superior as evidenced by the solution quality metric near 100% reported in Table 2. Convergence of the upper bound and the lower bound are indicated for the case of $\rho = 1.0$, $\zeta = 0.2$, and $\vartheta = 10$ under ‘high’ (base case) liquidity, in Figure 3. The solution convergence metric between two consecutive DCP upper bounding iterations is measured by $\frac{n||x^{(k+1)}-x^{(k)}||_2}{||x^{(1)}||_2}$.

### 3.2. Sensitivity on policy parameters

The preceding discussion reveals the adverse effects of elevated illiquidity and stricter policy parameters on the portfolio efficient frontier. Setting the risk aversion at a fixed level, $\vartheta = 10$, we study the expected portfolio return at the end of the month as policy pair $(\rho, \zeta)$ is varied. Market impact ex-ante model MIDO is used for this purpose, and this sensitivity at various levels of market illiquidity is presented in Figure 4. It is evident that as the target leverage $(\rho)$ and margin level $(\zeta)$ are both decreased, the impact on portfolio expected return is quite adverse. Also, note that for sufficiently-low leverage that the return is fairly insensitive to the margin policy for low or high liquidity levels.
4. Concluding Remarks

This paper presents a new approach to portfolio deleveraging incorporating margin and leverage constraints under price impact due to market illiquidity. The resulting model is a difficult non-convex optimization model involving quadratic objective and constraint functions. We proposed a novel dual cutting plane methodology to solve the Lagrangian upper bounding problem efficiently. Moreover, a lower bounding feasible portfolio generation algorithm is proposed so that the optimality of the deleveraged portfolio so-obtained can be ascertained.

Our deleveraging application using ETF assets demonstrates the validity and the efficacy of the approach we have taken in this paper. Furthermore, the portfolio analysis discussed in this paper provides insights into how policy parameters on portfolio margin/leverage must be considered when future uncertainties demand portfolio rebalancing to avoid portfolio catastrophe of the kind witnessed in past economic events.

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Finding centroid in DCP method and Market impact estimation

**EC.1. Theory of Lagrangian Dual Cutting-Planes (DCP)**

To develop our dual cutting-plane technique, consider (18) in a generalized setting as follows, i.e., the primal optimization problem:

\[
\max_{x \in X} \{ f(x) : g_j(x) \leq 0, j = 1, \ldots, m \}, \tag{EC.1}
\]

where \( X \subset \mathbb{R}^n \). We shall assume that (EC.1) is feasible. Other than the (once) differentiability, no particular properties are assumed for the functions \( f \) and \( g_j, j = 1, \ldots, m \). Define the Lagrangian of (EC.1) using the multipliers \( \pi_j, j = 1, \ldots, m \), of the inequality constraints:

\[
L(x, \pi) := f(x) - \sum_{j=1}^{m} \pi_j g_j(x). \tag{EC.2}
\]

The Lagrangian dual problem of (EC.1) is:

\[
L^* := \min_{\pi \in \Pi} L^*(\pi) \tag{EC.3}
\]

where

\[
L^*(\pi) := \max_{x \in X} L(x, \pi) \quad \text{and} \quad \Pi = \{ \pi \in \mathbb{R}^m : \pi_j \geq 0, j = 1, \ldots, m \}. \tag{EC.4}
\]

Feasibility of (EC.1) ensures that (EC.3) is bounded, i.e., at an optimal solution \( \pi^* \), \( L^{**} = L^*(\pi^*) > -\infty \), and thus, \( \pi_j^* < \infty, \forall j \). In determining the dual optimum value \( L^{**} \), the outer dual iteration of the min-max problem (EC.3) requires a computationally-intensive search over the unbounded region \( \Pi \) in (EC.4), for instance, using subgradient optimization (Bazaraa et al. 2006). Instead, in this paper, we propose an efficient scheme based on a sequence of dual cutting planes on a compact region that progressively shrinks the dual space to determine the dual optimum efficiently. To the best of our knowledge, such an approach has not been taken in the literature.

**EC.1.1. Mapping multipliers**

Consider the transformation (mapping) \( T = (T_1, \ldots, T_m) \) such that \( T_j : \mathbb{R}^m \to \mathbb{R} \), where:

\[
\theta_j = T_j(\pi) := \frac{\pi_j}{1 + \sum_{k=1}^{m} \pi_k}, \quad j = 1, \ldots, m. \tag{EC.5}
\]

**Proposition EC.1.** Suppose \( \pi \in \Pi \). Then, the following properties hold for \( T \):

(P1) \( \theta \in \Theta \) where the open set:

\[
\Theta := \{ \theta \in \mathbb{R}^m : \sum_{j=1}^{m} \theta_j < 1, \theta \geq 0 \}. \tag{EC.6}
\]
(P2) \( T \) has the unique inverse mapping, \( T^{-1}: \Theta \rightarrow \Pi \), given by:

\[
\pi_j = T^{-1}_j(\theta) := \frac{\theta_j}{1 - \sum_{k=1}^{m} \theta_k}, \quad j = 1, \ldots, m.
\]  

(EC.7)

On the open boundary of \( \Theta \), \( \pi_j \rightarrow \infty \), \( \forall j \); otherwise, \( \pi \) is finite.

(P3) \( T \) and \( T^{-1} \) are one-to-one and onto mappings with \( T(\Pi) = \Theta \) and \( T^{-1}(\Theta) = \Pi \).

Proof. Since \( \pi_j \geq 0 \), (EC.5) gives \( \theta_j \geq 0 \). Next, \( \sum_j \theta_j = \sum_j \pi_j > 1 \) since \( \sum_j \pi_j \geq 0 \), i.e., \( \theta \in \Theta \).

Next, rewriting (EC.5), for fixed \( \theta \in \Theta \):

\[
(1 - \theta_j)\pi_j - \sum_{k \neq j, k=1}^{m} \theta_j \pi_k = \theta_j, \quad j = 1, \ldots, m.
\]  

(EC.8)

To show that (EC.8) admits a unique solution in \( \Pi \), observe that (EC.8) is the system of equalities:

\[
(I - \theta 1^\top) \pi = \theta,
\]  

(EC.9)

where \( I - \theta 1^\top \) is non-singular for any \( \theta \in \Theta \). Applying Sherman-Morrison (1949) formula:

\[
(I - \theta 1^\top)^{-1} = I + \frac{\theta 1^\top}{1 - 1^\top \theta}
\]  

(EC.10)

which exists since \( 1^\top \theta < 1 \) due to \( \theta \in \Theta \). Therefore, the unique solution of (EC.5) for fixed \( \theta \in \Theta \) is:

\[
\pi = (I - \theta 1^\top)^{-1} \theta = \theta + \left( \frac{\theta 1^\top}{1 - 1^\top \theta} \right) \theta = \frac{\theta}{1 - 1^\top \theta} \in \Pi.
\]  

(EC.11)

Therefore, we have \( T^{-1}(T(\pi)) = \pi \) and \( T(T^{-1}(\theta)) = \theta \). Hence, there is one-to-one correspondence between \( \Pi \) and \( \Theta \). \( \square \)

EC.1.2. Dual reformulation

Define the function:

\[
\mathcal{L}(x, \theta) := (1 - \sum_{j=1}^{m} \theta_j)f(x) - \sum_{j=1}^{m} \theta_j g_j(x),
\]  

and consider the problem:

\[
\mathcal{L}^* := \min_{\theta \in \Theta} \mathcal{L}^*(\theta)
\]  

(EC.13)

where

\[
\mathcal{L}^*(\theta) := \max_{x \in X} \mathcal{L}(x, \theta).
\]  

(EC.14)
We shall derive a crucial relationship between $L^{**}$ in (EC.3) and $L^{**}$ in (EC.13). Let an optimal solution of the Lagrangian dual (EC.3) be $(\bar{x}, \bar{\pi})$. Define: $\bar{\theta} = T(\bar{\pi})$. Since $\bar{\pi} \geq 0$, from (P1), we have $\bar{\theta} \geq 0$ and $\sum_{j=1}^{m} \bar{\theta}_j < 1$, i.e., $\bar{\theta} \in \Theta$. Moreover, since $\bar{\pi} = T^{-1}(\bar{\theta})$,

$$L^{**} = L(\bar{x}, T^{-1}(\bar{\theta})) = f(\bar{x}) - \sum_{j=1}^{m} T^{-1}_j(\bar{\theta}) g_j(\bar{x})$$

$$= f(\bar{x}) - \sum_{j=1}^{m} \frac{\bar{\theta}_j}{1 - \sum_{k=1}^{m} \bar{\theta}_k} g_j(\bar{x})$$

$$= \frac{1}{1 - \sum_{j=1}^{m} \bar{\theta}_j} L(\bar{x}, \bar{\theta}). \tag{EC.15}$$

**Proposition EC.2.**

$$L^{**} = \min_{\theta \in \Theta} h(\theta) \quad \text{where} \quad h(\theta) := \frac{L^{**}(\theta)}{1 - \sum_{j=1}^{m} \theta_j}. \tag{EC.16}$$

**Proof.** Suppose $x^*(\theta)$ solves (EC.14) for fixed $\theta \in \Theta$. Then,

$$L^{**} \leq L(x^*(\theta), T^{-1}(\theta)) = f(x^*(\theta)) - \sum_{j=1}^{m} T^{-1}_j(\theta) g_j(x^*(\theta))$$

$$= f(x^*(\theta)) - \sum_{j=1}^{m} \frac{\theta_j}{1 - \sum_{k=1}^{m} \theta_k} g_j(x^*(\theta))$$

$$= \frac{L(x^*(\theta), \theta)}{1 - \sum_{j=1}^{m} \theta_j} = \frac{L^{**}(\theta)}{1 - \sum_{j=1}^{m} \theta_j}, \tag{EC.17}$$

and thus, $h(\theta) \geq L^{**}$ for any $\theta \in \Theta$. Next, it is only necessary to observe that there exists $\bar{\theta} \in \Theta$ and $\bar{x}(\bar{\theta}) \in X$ such that $h(\bar{\theta}) = L^{**}$ is held due to (EC.15). □

Therefore, instead of solving the original dual problem in (EC.3), we may solve the equivalent formulation in (EC.16), in which $h(\theta)$ is defined over a compact (open) set $\Theta$. We develop an iterative cutting plane technique on $\Theta$ so that (EC.16) can be computed efficiently.

**EC.1.3. Dual cutting-plane (DCP) method**

Suppose at some iteration $t$, given $\theta^t \in \Theta$, solve the dual subproblem given by (EC.14), i.e.,

$$L^*(\theta^t) := \max_{x \in X} L(x, \theta^t) = (1 - \sum_{j=1}^{m} \theta^t_j) f(x) - \sum_{j=1}^{m} \theta^t_j g_j(x). \tag{EC.18}$$

Let a global optimal solution of (EC.18) be $x^t \equiv x^*(\theta^t)$, i.e.,

$$L^*(\theta^t) = L(x^t, \theta^t) = (1 - \sum_{j=1}^{m} \theta^t_j) f(x^t) - \sum_{j=1}^{m} \theta^t_j g_j(x^t). \tag{EC.19}$$

The next iterate $\theta^{t+1} \in \Theta$ must be chosen such that $h(\theta^{t+1}) < h(\theta^t)$ toward solving the minimization in (EC.16); however, since $x^t \in X$ is feasible in (EC.14) for $\theta^{t+1}$, we must have:

$$L^*(\theta^{t+1}) \geq L(x^t, \theta^{t+1})$$
which yields
\[
\begin{align*}
    h(\theta^{t+1}) &= \frac{L^*(\theta^{t+1})}{1 - \sum_{j=1}^{m} \theta_j^{t+1}} \\
    &\geq \frac{L(x^t, \theta^{t+1})}{1 - \sum_{j=1}^{m} \theta_j^{t+1}} = f(x^t) - \sum_{j=1}^{m} \frac{\theta_j^{t+1} g_j(x^t)}{1 - \sum_{k=1}^{m} \theta_k^{t+1}}.
\end{align*}
\]  
(EC.20)

**Proposition EC.3.** A new iterate \( \theta \) cannot lead to an improvement in the objective value over \( h(\theta^t) \) unless the following linear inequality holds:
\[
\sum_{j=1}^{m} \alpha_j^t \theta_j > \beta^t,
\]  
(EC.21)

where the coefficients \( \alpha^t \in \mathbb{R}^m \) and \( \beta^t \in \mathbb{R} \) are given by:
\[
\alpha_j^t := g_j(x^t) + \beta^t, \quad j = 1, \ldots, m \quad \text{and} \quad \beta^t := f(x^t) - h(\theta^t) = \frac{\sum_{j=1}^{m} \theta_j^t g_j(x^t)}{1 - \sum_{j=1}^{m} \theta_j^t}.
\]  
(EC.22)

**Proof.** Combining \( h(\theta^{t+1}) < h(\theta^t) \) with (EC.20), for \( \theta \equiv \theta^{t+1} \):
\[
\begin{align*}
    f(x^t) - \sum_{j=1}^{m} \frac{\theta_j}{1 - \sum_{k=1}^{m} \theta_k} g_j(x^t) < h(\theta^t),
\end{align*}
\]
which yields:
\[
\begin{align*}
    \sum_{j=1}^{m} \theta_j g_j(x^t) > (f(x^t) - h(\theta^t)) \left[ 1 - \sum_{j=1}^{m} \theta_j \right]
\end{align*}
\]
or,
\[
\begin{align*}
    \sum_{j=1}^{m} \left[ g_j(x^t) + f(x^t) - h(\theta^t) \right] \theta_j > f(x^t) - h(\theta^t).
\end{align*}
\]

Noting (EC.22), the proof is completed. \( \square \)

Observe that for \( \theta = \theta^t \), we have \( \sum_{j=1}^{m} \alpha_j^t \theta_j = \beta^t \). Thus, the dual cutting plane generated on \( \Theta \) at iteration \( t \) is: \( \alpha^T \theta \leq \beta^t \). Hence, the updated (shrunk) dual feasible region is:
\[
\Theta_t := \Theta \setminus \bigcup_{\tau=1}^{t} \left\{ \theta \in \mathbb{R}^m : \alpha^\tau \theta \leq \beta^\tau \right\}.
\]  
(EC.23)

While \( \Theta \) is an \( m \)-dimensional simplex (with apex at the origin), observe that \( \Theta_t \) can be a general polytope in \( m \)-dimensions. The next iterate \( \theta^{t+1} \) is set to be the centroid of \( \Theta_t \), denoted by \( \theta^{t+1} = C(\Theta_t) \) for \( t = 0, 1, 2, \ldots \), where we have set \( \Theta_0 \equiv \Theta \); see the two-dimensional illustration in Figure EC.1. The termination criterion for improving the (dual) solution of (EC.16) is when \( ||\theta^{t+1} - \theta^t|| < \varepsilon \) for a specified (scaled) tolerance \( \varepsilon > 0 \). To determine the centroid of \( \Theta_t \), we employ a standard technique, e.g., Cut-Off Polyhedron (COP) method (Nakagawa et al. 1984), see Appendix A.
EC.2. Theory of Feasible Portfolio Generation

The application of the preceding DCP method on the deleveraging problem (18) yields its optimal Lagrangian dual value $L^{**}$ in (EC.3), associated with an optimal primal-dual pair, $(x^*, \pi^*)$. Then, $L^{**}$ is an upper bound on (18), denoted by $F_U(\rho, \zeta, \vartheta)$, that is, $F_U(\rho, \zeta, \vartheta) \geq F(\rho, \zeta, \vartheta)$. If $x^*$ is feasible in (18), and the complementary conditions $\pi^*_j g_j(x^*) = 0$ are satisfied for all $j$, then $x^*$ indeed is a global optimal portfolio solution; otherwise, there may exist a nonzero duality gap for (18). In this case, we generate an improving-sequence of lower bounds, starting with the solution estimate $x_U$ such that a tight lower bound $F_L(\rho, \zeta, \vartheta) \leq F(\rho, \zeta, \vartheta)$ is available with a feasible portfolio solution $x_L$ of (18).

Toward this, consider MIDO model in (18) in the following stylized format of a quadratic program with an objective function having a separable indefinite part, and a convex nonseparable part $h(x)$ that represents the portfolio variance risk function $\frac{1}{(w_0)^2} x^T P_0 V P_0 x$, along with $m$ indefinite, but separable, quadratic constraints and simple bounds on variables:

$$Z^* = \max_{l \leq x \leq u} f(x) \equiv \sum_{j \in J_0^+} d_{0j} (x_j)^2 + \sum_{j \in J_0^-} d_{0j} (x_j)^2 + c_0^T x - h(x)$$

s.t. \( \sum_{j \in J_i^+} d_{ij} (x_j)^2 + \sum_{j \in J_i^-} d_{ij} (x_j)^2 + c_i^T x \leq b_i, \quad i = 1, \ldots, m \). (EC.24)

Denoting the diagonal elements of the variance-covariance matrix $V$ by $\sigma_j^2$, $j = 1, \ldots, n$, we have defined in (EC.24) the following index subsets:

$$J_0^+ := \left\{ j = 1, \ldots, n \mid d_{0j} > \vartheta \left( \frac{p_{0j} \sigma_j}{w_0} \right)^2 \right\}$$

and $J_0^- := \{ 1, \ldots, n \} \setminus J_0^+$; moreover, for $i = 1, \ldots, m$, define $J_i^- := \{ j = 1, \ldots, n \mid d_{ij} < 0 \}$ and $J_i^+ := \{ 1, \ldots, n \} \setminus J_i^-$. Note that the format in (EC.24) resembles the MIDO model in (18).
At the \( k \)th iteration of generating a lower approximation, let \( x_1 = y^k \) be a given portfolio solution. at the initial approximation \((k = 0)\), we set \( y^0 = x_1^L \). Using Taylor’s first-order approximation (of indefinite terms) of \((EC.24)\) at \( y^k \):

\[
\sum_{j \in J^+_0} d_{0j}(x_j)^2 \geq \sum_{j \in J^+_0} [d_{0j}(y_j^k)^2 + 2d_{0j}y_j^k(x_j - y_j^k)] = \sum_{j \in J^+_0} 2(d_{0j}y_j^k)x_j - \sum_{j \in J^+_0} d_{0j}(y_j^k)^2
\]

and for \( i = 1, \ldots, m \):

\[
\sum_{j \in J^-_i} d_{ij}(x_j)^2 \leq \sum_{j \in J^-_i} [d_{ij}(y_j^k)^2 + 2d_{ij}y_j^k(x_j - y_j^k)] = \sum_{j \in J^-_i} 2(d_{ij}y_j^k)x_j - \sum_{j \in J^-_i} d_{ij}(y_j^k)^2.
\]

Then, consider the following concave maximization problem:

\[
f_{\ell}(y^k) := \max_{\|x\| \leq c_0} \sum_{j \in J^+_0} d_{0j}(x_j)^2 + c_0^T x + 2 \sum_{j \in J^+_0} (d_{0j}y_j^k)x_j - h(x)
\]

s.t. \( \sum_{j \in J^+_i} d_{ij}(x_j)^2 + c_i^T x + 2 \sum_{j \in J^+_i} (d_{ij}y_j^k)x_j \leq b_i + \sum_{j \in J^-_i} d_{ij}(y_j^k)^2, \quad i = 1, \ldots, m. \) (EC.27)

Then, it is straightforward to claim that \( Z^* \geq f_{\ell}(y^k) - \sum_{j \in J^+_0} d_{0j}(y_j^k)^2 \) since the inequalities (EC.26) ensure that the feasible region of (EC.27) is a ‘restriction’ over that of (EC.24) and the objective of the latter is a lower approximation on the former. Moreover, (EC.27) can be solved to global optimality using standard convex programming methods. In fact, the DCP method presented earlier may be employed where the corresponding DCP subproblem (EC.18) is solved to global optimality, which indeed is a global optimum for (EC.27) since strong Lagrangian duality holds for convex programs. Let a (global) optimum solution of (EC.27) be denoted by \( y^{k+1} \). The fact that a monotonic sequence of lower bounds can be generated is claimed in the following result.

**Proposition EC.4.** Suppose the fixed \( y^k \) is chosen feasible in \((EC.24)\), and let an optimal solution of \((EC.27)\) be denoted by \( y^{k+1} \). Then, \( y^{k+1} \) is feasible in \((EC.24)\), and \( Z^* \geq f(y^{k+1}) \geq f(y^k) \).

Moreover, \( f(y^{k+1}) - f(y^k) \geq \sum_{j \in J^+_0} d_{0j}(y_j^{k+1} - y_j^k)^2 \geq 0 \).

**Proof.** First, given \( y^k \) is feasible in (EC.24), and since the first-order approximation is performed at \( y^k \) to obtain the constraints of (EC.27), \( y^k \) must also be feasible in (EC.27). Then, due to optimality of \( y^{k+1} \) in (EC.27),

\[
f_{\ell}(y^k) = \sum_{j \in J^+_0} d_{0j}(y_j^k)^2 + c_0^T y^k + 2 \sum_{j \in J^+_0} (d_{0j}y_j^k)y_j^{k+1} - h(y^{k+1}) \geq \sum_{j \in J^+_0} d_{0j}(y_j^k)^2 + c_0^T y^k + 2 \sum_{j \in J^+_0} (d_{0j}y_j^k)y_j^k - h(y^k) = f(y^k) + \sum_{j \in J^+_0} d_{0j}(y_j^k)^2.
\]
Next,
\[ f(y^{k+1}) = \sum_{j \in J_0^+} d_{0j} (y_j^{k+1})^2 + \sum_{j \in J_0^-} d_{0j} (y_j^{k+1})^2 + c_0^T y^{k+1} - h(y^{k+1}) \]
\[ = f_L(y^k) - 2 \sum_{j \in J_0^+} (d_{0j} y_j^k) y_j^{k+1} + \sum_{j \in J_0^-} d_{0j} (y_j^k)^2 \]
\[ \geq f(y^k) + \sum_{j \in J_0^+} d_{0j} (y_j^k)^2 - 2 \sum_{j \in J_0^+} (d_{0j} y_j^k) y_j^{k+1} + \sum_{j \in J_0^-} d_{0j} (y_j^k)^2 \]
\[ = f(y^k) + \sum_{j \in J_0^+} d_{0j} (y_j^{k+1} - y_j^k)^2 \]
\[ \geq f(y^k), \]
where the first inequality follows due to (EC.28), and the second inequality holds due to \((y_j^{k+1} - y_j^k)^2 \geq 0\) and \(d_{0j} > 0, j \in J_0^+\). This completes the proof. □

Therefore, at some iteration \(k\) when \(y^k\) is feasible in (EC.24), the convex model (EC.27) not only generates an solution \(y^{k+1}\) feasible in (EC.24), but also it provides an improved lower bound, i.e., \(f(y^{k+1}) > f(y^k)\), provided at least one component of \(y_j^{k+1}\) is different from \(y_j^k\) for some \(j \in J_0^+\). That is, so long as the solution of (EC.27) varies iteratively, a strictly-improving sequence of lower bounds on (EC.27) is generated. The sequence must converge since the optimal value \(Z^*\) of (EC.24) provides a finite upper bound on the monotonic sequence \(\{f(y^k)\}\); moreover,

**Proposition EC.5.** Suppose an optimal solution of (EC.24) is denoted by \(x^*\), i.e., \(Z^* \equiv f(x^*)\). Let an optimal solution of (EC.27) with \(y^k = x^*\) be given by \(x^{**}\). Then, \(f(x^{**}) = f^*\), i.e., \(x^{**}\) solves (EC.24).

**Proof.** Due to Proposition EC.4, we must have \(f(x^{**}) \geq f(x^*)\). However, since \(x^{**}\) is feasible in (EC.24), \(f(x^{**}) \leq f^*\) holds, and thus, it follows that \(f(x^{**}) = f^*\). □

**Remark:** For \(k = 0\), \(y^0 = x'_{\ell} \) will be infeasible in (EC.24), for if not, \(x'_{\ell} \) is an optimal deleveraged portfolio in (18). Suppose the resulting model (EC.27) for the lower bound \(f_L(y^0)\) is also infeasible. Since the fully-liquidated portfolio \(x_1 = 0\) is feasible in (EC.24), see Proposition 1, in this case, we set \(y^0 = 0\), so that (EC.27) is guaranteed to be feasible.

On the other hand, suppose the model (EC.27) resulting from \(y^0 = x'_{\ell} \) is feasible, but its optimal solution \(y^1\) is infeasible in (EC.24). In this case, we reset \(y^0 = 0\) so that the optimal solution \(y^1\) is guaranteed to be feasible in (EC.24).

**EC.3. Appendix A: Finding the centroid of \(\Theta_t\)**

We employ the COP (Cut-Off Polyhedron) method in Nakagawa et al. (1984) to determine the centroid of a polyhedron defined by a finite set of hyperplanes. We present the basic elements briefly here by adapting and modifying to our dual cutting plane context.
The initial polyhedron in (EC.6), referred to as $\Theta_0$, is formed by \( m + 1 \) (facet-inducing) hyperplanes and \( m + 1 \) vertices, given by \( m \) the elementary vectors \( e_1, \ldots, e_m \) and the origin \( 0 \). After \( t \) iterations of the DCP technique, the resulting polyhedron \( \Theta_t \) in (48) is determined by \( K_t \) hyperplanes (after removing the non-facet-inducing and redundant hyperplanes from the total of \( m + 1 + t \)), and its set of extreme points is denoted by \( V_t = \{ v^r : r = 1, \ldots, R_t \} \). Note that both \( K_t \) and \( R_t \) are non-monotonic in \( t \), and they have a minimum value of \( m + 1 \). For the remainder of this section, we shall drop the iteration count sub (super-)script \( t \) for ease of exposition.

Let \( Y \) be a logical (0-1) matrix that indicates which of the \( K \) hyperplanes are connected by which of the vertices in \( V \) to form the matrix \( Y \in \mathbb{R}^{R \times K} \), where its \((r,k)\)th element:

\[
[Y]_{r,k} \equiv \begin{cases} 
1 & \text{if a vertex } v^r \text{ is on the } k \text{th hyperplane} \\
0 & \text{otherwise.}
\end{cases}
\]

For the initialization step with \( \Theta_0 \), the logical matrix \( Y_0 = U - I \in \mathbb{R}^{(m+1) \times (m+1)} \), where for the square matrix \( U \) all elements are unity, i.e., \( [U]_{i,j} = 1 \), and \( I \) is the \((m+1)\)-dimensional identity matrix.

Note that \( C(\Theta) = 1/R \sum_{r=1}^{R} v^r \), and thus, the dual cutting plane satisfies \( \alpha^\top C(\Theta) = \beta \). For the polyhedron updated by the dual cut, i.e., \( \Theta \cap \{ \alpha^\top \theta \geq \beta \} \), the updated set of vertices is required. Then, the following operations are performed.

Step 1: Determine the index sets of feasible and infeasible vertices of \( \Theta \):

\[
R^- = \{ r : \alpha^\top v^r < \beta \}, \quad R^0 = \{ r : \alpha^\top v^r = \beta \}, \text{ and } R^+ = \{ r : \alpha^\top v^r > \beta \},
\]

where \( R^- \) is the set of indices for infeasible vertices under the dual cut, \( R^0 \) is the set of indices for vertices that are on the dual cut, and \( R^+ \) is the set of indices for vertices that are strictly feasible under the dual cut. Note that \( R^+ \neq \emptyset \) since \( C(\Theta) \) is in the relative interior of \( \Theta \) and \( \alpha^\top C(\Theta) = \beta \).

Step 2: Find planes that will form a new \( \Theta \).

\[
K^+ = \left\{ k : \sum_{r \in R^+} [Y]_{r,k} > 0, \; k = 1, \ldots, K \right\}
\]

Step 3: Find new vertices of a new \( \Theta \). The new vertices are generated where the dual cut meets edges made of a vertex in \( R^+ \) and a vertex in \( R^- \). Thus, for all pairs of \( r^+ \in R^+ \) and \( r^- \in R^- \), do the following sub-steps:

- **Step 3.1:** Get a 0-1 vector \( t \in \mathbb{R}^{|K^+|} \) that indicates whether the a plane formed by \( v^{r^+} \) and \( v^{r^-} \):

\[
t = [Y]_{r^+,k} \cdot [Y]_{r^-,k} \text{ for } k \in K^+
\]
Step 3.2: If $\sum_{j=1}^{[t]} t_j = m - 1$, then, since $v^r_+$ and $v^r_-$ are on the same edge, do the following two sub-steps:

- Update $R \leftarrow R + 1$, $R^0 \leftarrow R^0 \cup \{R\}$, and $[Y]_{R,k} = t^\top$ for $k \in K^+$.
- Find a new vertex $v^R = v^r_+ + \sigma(v^r_- - v^r_+)$, where the step size $\sigma$ is

$$\sigma = \left( \beta - \alpha^\top v^r_+ \right) / \left( \alpha^\top [v^r_- - v^r_+] \right).$$

Step 4: Add a new column on $Y$ for the hyperplane of the dual cut.

- Set $Y_{r,K+1} = 1$ for $r \in R^0$ and $Y_{r,K+1} = 0$ for $r \in R^+$

Step 5: Reconstruct $Y$ and $V$ for the new polyhedron $\Theta$

- $R = |R^+| + |R^0|$, $K = |K^+| + 1$, $V = \{v^r : r \in R^+ \cup R^0\}$
- $Y = [Y]_{r,k}$ for $r \in R^+ \cup R^0$ and $k \in K^+ \cup \{K\}$

Step 6: Get the centroid $C(\Theta) = \frac{1}{R} \sum_{r=1}^{R} v^r$.

EC.4. Appendix B: Estimating Market Impact Parameters

Market impact parameters are estimated using TAQ (Millisecond Trade and Quote) data from NYSE. The trade (execution) transactions data for a given day (for given stock or ETF) are aggregated into five-minute intervals during regular trading time. Then, the total of 390 minutes per day (from 09:30-16:00 hrs) is divided into $T = 78$ five-minute intervals, for each trading day in the sample of $N$ number of days. For a given day $d$, at some time $t$, the price of a (generic) asset is denoted by $p_{t,d}$. Hence, the open price for the day is $p_{0,d}$. By classifying each trade using the tick rule in Asquith et al. (2009), the following trade statistics are calculated for each five-minute interval (denoted $t$) in the day $d$:

1. Total trade volume: $v_{t,d}$ (shares)
2. Net trade volume: $\hat{v}_{t,d}$
3. Dollar trading volume: $s_{t,d}$
4. Open and close prices (in the 5-min interval): $p^o_{t,d}$ and $p^c_{t,d}$
5. Given an interval $(t,d)$, let there be $n$ distinct trades. The price change of two consecutive trades is denoted $\delta p_i$, if greater than $0.005$, otherwise, $0$. The sum of absolute values of price changes in the 5-min interval: $\Delta p(t,d) = \sum_{i=1}^{n} |\delta p_i|$

---

8 The net volume of a stock is a consolidated total of the positive and negative movements of the security over the period, i.e., up-tick volume minus its down-tick volume in the 5-min interval. A positive (negative) net volume indicates greater upward (downward) movement associated with net ‘buying’ (‘selling’) in the security over the five minutes.

9 The dollar volume is computed by multiplying each trade size by the execution price of the trade, and summing up over all trades in the 5-min interval.
The total net volume traded in a day until time $t$ is $\sum_{\tau=1}^{t} \hat{v}_{\tau,d}$. This leads to the permanent price change by the end of period $t$ given by $(p_{c,t,d} - p_{0,d})$, the price differential at the end (close) of the time period and the price at the beginning of the day. On the other hand, the temporary impact is due to trading at higher rates causing a temporary lack of liquidity to absorb the required trading rate.

Given a 5-min time period, the temporary price effect is measured by the product of the level of illiquidity in the period, $I_{L,t,d}$, and the (effective) trade price against illiquidity, that is, $I_{L,t,d} \times p_{vw,t,d}$, where volume-weighted average price (VWAP) is used for the effective price. We employ the most widely liquidity measure developed by Amihud (2002), and define:

$$I_{t,d} = \frac{\Delta p(t,d)}{s_{t,d}}$$

as the measure of illiquidity during a 5-min interval. Then, the temporary effect is

$$I_{L,t,d} \times p_{vw,t,d} = \frac{\Delta p(t,d)}{s_{t,d}} \times \frac{s_{t,d}}{v_{t,d}} = \frac{\Delta p(t,d)}{v_{t,d}}.$$  \hspace{1cm} (EC.30)

Combining the permanent and temporary price effects, we estimate the model:

$$(p_{c,t,d} - p_{0,d}) + \Delta p(t,d) = \gamma \sum_{\tau=1}^{t} \hat{v}_{\tau,d} + \lambda v_{t,d} + \varepsilon_{t,d}, \quad t = 1, \ldots T, \quad d = 1, \ldots N.$$  \hspace{1cm} (EC.31)

References
