Investors expect more than is commonly assumed, and it matters.

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Abstract

We demonstrate how momentum expectations can be quantified ex-ante by using option’s implied volatility surface with allowance for autocorrelation. Then, together with ex-ante tail-risk, we empirically investigate how corresponding believes relate to future realized returns. Analyzing the U.S. cross-section and the SP500 Index, ex-ante momentum positively predicts equity performance while ex-ante tail risk causes negative alphas, both effects being highly significant and economically large. Thus, also from an risk-neutral perspective, momentum anomalies and distress puzzles are observable in the U.S. market. Robustness of the results is confirmed from different test settings and consistency over time.

Keywords: Risk-Neutral Moments, Implied Volatility, Momentum, Skewness, Kurtosis, Fractal Brownian Motion

JEL: G11, G12
1 Introduction

The equilibrium of expected risk to return is fundamental in asset pricing. However, expectations themselves are difficult to quantify, hence the discipline’s classical approach is to analyze the relation between historical risk patterns to future realized returns. Our objective is to release typical model assumptions of un-autocorrelated and normally distributed returns and evaluate a framework, of how respective expectations can be measured ex-ante before they are actually realized. Doing so, we address recent debates about risk-neutral measurement of equity risk factors. While empirical effects of ex-ante skewness and kurtosis already enjoy an academic discussion\textsuperscript{1}, we see our concept of ex-ante momentum to be a new contribution to academic literature.

Let risk expectations be fully reflected within the shape of the ex-ante return distribution, including (all) higher moments. Additionally, existence of momentum implies that (physical) autocorrelation is non-zero (Jegadeesh & Titman (1993)) such that returns are drifted. Thus, to bring classical momentum literature into an ex-ante setting, if there exist expectations on momentum, then investors believe in an drift of expected returns. So to say, we hypothesize that releasing zero-autocorrelation and normality assumptions of basic portfolio theory (Markowitz (1952)) gives a picture coming closer to reality - expected risk as volatility plus higher moments and expected returns consisting of un-drifted and autocorrelated components. Applying arbitrage free principles and efficient market hypothesis (Fama (1970)), all available information at time $t$ is priced in, therefore current investor believes are not only represented at the stock price, but also in related option prices. Options have the advantage that they are per definition forward looking, thus provide risk-neutral\textsuperscript{2} information (c.p. Latané & Rendleman (1976)). They further allow to observe an entire surface of expectations at time $t$ with several levels of strikes and times to maturities, different to stocks having only one single data point of price per $t$.

\textsuperscript{1}E.g. Cremers & Weinbaum (2010), Schneider et al. (2016) or Bali, Hu & Murray (2017) among others.

\textsuperscript{2}We use the wording ‘risk-neutral’ in the sense that risk expectations are directly observed from actually traded prices/volatilities, thus reflect real investors’ beliefs. Therefore, no assumptions about corresponding utility functions have to be made, as risk aversions are already embedded within actual prices/data, making option implied volatilities risk-neutral. Our wording of risk-neutral implies that information is ex-ante, since it is the expectation on a pattern before it is actually physically realized.
That empirically, the option implied volatility surface has significant impact on future realized portfolio returns is earlier discussed in Bali & Hovakimian (2009) or Banerjee et al. (2007). Since then, further research like Mixon (2010), Bali, Hu & Murray (2017) or Schneider et al. (2016) shows how ex-ante skewness and kurtosis can be estimated, typically by fixing time to maturity and fitting observable Black & Scholes (1973) (B/S in the following) volatilities over degrees of moneyness. As literature in this field comes to contrary conclusions how ex-ante skewness/kurtosis relates to future returns, we analyze corresponding empirical effects on our own. Similar to Cremers & Weinbaum (2010), DeMiguel et al. (2013) and Schneider et al. (2016), we observe that ex-ante skewness positively relates to future portfolio returns. In contrast to Bali, Hu & Murray (2017), we find higher ex-ante kurtosis to be followed by worse performance. However, we claim that our empirical results are consistent with economic theory. As negative skewness and positive kurtosis lift likelihood of extreme adverse returns, we refer this pattern to as tail risk. Via Merton (1974)'s credit risk model, negatively tailed distributions push probability of default, thus in our opinion, skewness and kurtosis effects on returns have to be in opposite directions, which contradicts empirical patterns of Bali, Hu & Murray (2017). In our setting, ex-ante tail-risk causes significant under-performance. At analysis of the U.S. cross section we observe that the high-minus-low ex-ante tail-risk portfolio realizes a yearly Fama & French (2015) alpha of -9.2% with low exposure to the risk factors (within ±0.3) and weak $R^2$ of 15.4%. From investment timing of the SP500, above cumulative average tail-risk induces alpha of -4.8% p.a measured on five days forward returns. When considering the link between distribution tails and credit risk, with these findings we confirm existence of a distress puzzle (e.g. Campbell et al. (2008)) from an risk-neutral point of view.

Beyond contributing to the academic discussion about ex-ante skewness/kurtosis effects, we propose a new framework of how momentum expectations can be quantified ex-ante from option data, having stock’s expected payoff structure replicated by ATM options. Further,
connecting the wording momentum to autocorrelation in returns, for risk-neutral purposes it is thus crucial to allow autocorrelation in the stochastic process under which options are priced. We do so by building on the option pricing model of Hu & Øksendal (2003), where standard Brownian motion is replaced by (possibly autocorrelated) fractal Brownian motion and examine the respective B/S-volatility surface over times-to-maturities. Empirically, implied volatilities show strong curvatures which are fit well by fractal option pricing, e.g. $R^2$ ranging between 94.5% and 99.7% in the case of the SP500. The slope of this curvature is deterministic for expected momentum measurement. Within investment timing at the SP500, we see a betting on ex-ante momentum strategy being able to realize an alpha of 4.7% p.a. before costs. On a cross sectional level, we also observe an positive tendency between stock’s ex-ante momentum believes and portfolio performance, here the difference between high to low momentum stocks is an alpha of 4.8% p.a., also with very small risk factor loadings ($\pm 0.36$) and small $R^2$ of 16.7%. In one sentence, we observe throughout empirical evidence for pricing anomalies in the U.S. market which are sourced in investor expectations that are beyond commonly made assumptions. Figure 1 summarizes ex-ante momentum and tail-risk effects on future realized returns within the cross section, it can be seen that the effects are of relevant size and consistency over quantiles.

41st and 3rd quartile on daily regression fit of fractal pricing on SP500 options, details can be found in Section 2.
Figure 1
U.S. cross section sorted on risk-neutral characteristics from low to high. Alphas presented as risk-adjusted performance yearly, measured upon daily portfolio returns using Fama & French (1993) three factor model. We find two significant and consistent patterns of investor expectations to future returns: (i) ex-ante momentum positively predicts future realized alpha and (ii) ex-ante tail-risk negatively relates to portfolio performance. Expectations are quantified from option implied volatility surfaces. For both cases, the observed effects show consistency over quantiles.

With an short excursus on volatility prediction, we apply machine learning on SP500 data to show that option implied momentum and tail risk are valuable candidates at this task, e.g. achieving an out-of-sample $R^2$ of 64.4% for one-week-ahead SP500 volatility prediction.

The remainder of this paper is set up as follows. Section 2 shows the theoretical model of our ex-ante momentum measure, provides economic argumentation for it and introduces the risk-neutral skewness/kurtosis estimates used. After the theoretical settings, we continue at Section 3 and data used. Empirical analysis are then made first on a market level (SP500; Section 4) and on the cross section of U.S. equity (Section 5). Section 6 concludes.

2 Measuring Expectations

To overcome backward looking bias, we build our research on a forward looking basis using actually traded investor expectations. Accordingly, we take option data which are per contract definition priced on the underlying’s believed risk-return trade-off. Different to stocks, where one can only observe one price for a point in time, respective options for the same point in time come with a variety of prices differing in time-to-maturity and strike levels.
Hence instead of one dimension of price, options allow to draw a multidimensional surface out of investor expectations. As option prices, maturities, strikes and the risk free rate are direct observable from trading data, option pricing formulas can be reformulated to measure the volatility used for pricing, which gives the option implied volatility surface. We generally take Black & Scholes (1973) option implied volatility surface from European type options - which per model notation should be flat in all dimension - and decompose it over times-to-maturities (aka. term structure or time dimension of variance) and levels of strikes (distributional dimension, shape of variance). We see it crucial to take implied volatilities under Black-Scholes option pricing as this framework is built on very similar assumptions as CAPM (Sharpe (1964), Lintner (1965), Mossin (1966)) and other common asset pricing models (e.g. Carhart (2012), Fama & French (1993, 2015)) such as normally distributed returns and geometric Brownian motion having uncorrelated increments(= no momentum; except Carhart (2012) who account for momentum but not for skewness/kurtosis). From non-constant implied volatility surfaces we derive connections to expected momentum and tail risk.

Risk-Neutral Momentum Taking ATM options, a stock’s expected payoff can be replicated by 1:1 relation of Puts and Calls, for which’s implied volatility surface is modeled to be flat under Black-Scholes - empirically this is hardly the case (see Fig. 2 and 7). Fixing the strike level to at-the-money, Figure 2 shows the empirical B/S volatility surface over time to maturity of the SP500 for two different points in time (October 1st, 2008 and October 1st, 2018).
Figure 2
Annualized empirical Black-Scholes implied volatility surface of ATM options on the SP500 for two different points in time (Oct. 1\textsuperscript{st} 2008 and Oct. 1\textsuperscript{st} 2018). In 2018, the volatility curve is upward sloping meaning that investors expected less uncertainty in the near future than many months ahead. In times of the financial crisis the picture is other way around, the downward sloping trend indicates that investors expect a decrease in volatility of the U.S. market over the long run. Strike is set to ATM such that Call prices equal the Puts, hence expectations are isolated from skew and smirk effects implied by OTM volatilities. This shows, that for the same stock (replicated by Puts and Calls), there exists an (typically non-flat) term structure of expected risk.

In 2018, we see that investors expected an upward sloping trend in traded volatility, meaning that the near future is believed to be less risky than the future farther away. The picture of 2008 looks very different, in the middle of the financial crisis investors traded high volatility levels for the few months ahead, but expected a decrease in volatilities over the long run, which leads into a downward sloping volatility curve. However, under Black-Scholes assuming common Brownian motion, the theoretical volatility curve is flat, which led us to the suggestion, that this implied volatility curve covers investor expectations about future price development.

In order to isolate momentum expectations from skew and smirk effects of OTM implied volatilities, we set strike prices equal to ATM such that Call prices equal Put prices, replicating the underlying’s payoff structure. In the presence of momentum believes, the movement of the underlying is no longer independent in its increments thus standard Brownian motion and the assumption of zero autocorrelation have to be rejected. At this point, we introduce
the stochastic concept of Hurst (1956)’s exponent and fractal Brownian motion (Mandelbrot & Van Ness (1968)). In the following of our work, the Hurst exponent $H$ takes on a crucial role for indicating risk-neutral momentum believes. Ever since, various estimation methods were developed to measure the Hurst exponent, including rescaled range (R/S; Hurst (1956), Mandelbrot & Van Ness (1968)), modified R/S (Lo (1991)), wavelet (Veitch & Abry (1999)), rescaled variance (Giraitis et al. (2003)) and several other methods, Rea et al. (2009) provide an empirical comparison of 12 different such frameworks. What all those methods have in common - which in our opinion also explains their weak predictability of future patterns - is that they are measured physically, on a historical basis. Hu & Øksendal (2003) and Elliott & Van Der Hoek (2003) provide the corresponding forward looking estimation method for the Hurst exponent via fractional Black-Scholes option pricing. A detailed methodological description can be found in Li & Chen (2014). Other relevant literature addressing option implied measurement of the Hurst exponent and the related empirical patterns are for example Flint & Mare (2016) or Funahashi & Kijima (2017). Starting point for the risk neutral-measurement of $H$ is given by Mandelbrot & Van Ness (1968), who model long-term memory into the standard Brownian motion process such that increments do not necessarily have to be independent any longer. While the expectation of this continuous-time Gaussian process $B_H(t)$ is zero at every point of time,

$$E[B_H(t)] = 0$$  \hfill (1)$$

the main difference between standard to fractal Brownian motion comes from its covariance function

$$E[B_H(t), B_H(s)] = \frac{1}{2} [t^{2H} + s^{2H} - |t-s|^{2H}],$$ \hfill (2)

making it a self-similar process with stationary increments and $H$ as the so called Hurst exponent. Importantly, $H$ is essential as it determines the form of the stochastic process. The Hurst exponent $H$ ranges from 0 to 1, where $H > 0.5$ indicates positive autocorrelation in the increments. Under $H < 0.5$, increments are negatively correlated and a $H$ of exactly 0.5 results in no autocorrelation, such that we derive standard Brownian Motion again. Figure 3 displays sample paths of fractal Brownian motions under different Hurst exponents
$H$, representing negative-, no- and positive long-term memory.

**Figure 3**
Simulated paths of fractal Brownian motions with negative correlation in increments ($H < 0.5$), perfect Brownian motion ($H = 0.5$) and positive autocorrelation ($H > 0.5$). As the paths show, $H < 0.5$ causes paths with trend reversals, $H = 0.5$ no trend and $H > 0.5$ trend continuation in the stochastic process.

Hu & Øksendal (2003) take this fractal Brownian motion to model the price process of a risky stock $S$ over time having

$$\frac{dS_{t+\tau}}{S_t} = \mu dt + \sigma dB_H(t)$$

with $\mu$ as the drift parameter and $\tau = T - t$ as the time to maturity. Following, they show that the variance in the underlying’s price given fractal Brownian motion can be expressed by

$$\text{var} \left[ \ln \left( \frac{S_{t+\tau}}{S_t} \right) \right] = \sigma_f^2 \tau^{2H},$$

with $\sigma_f$ as the fractal volatility, which is simply the level of classical volatility $\sigma$ corrected by the Hurst exponent $H$. Note that in the case of $H = 0.5$, fractal volatility $\sigma_f$ equals classical volatility again. Implementing this fractal process into standard Black-Scholes option pricing formula, the price of an European call option under non-zero autocorrelation $C_f$ is defined as

$$C_f(S_t, K, \tau, r, \sigma, H) = S_t \ast \phi(\hat{d}_1) - K^{-\tau} \ast \phi(\hat{d}_2),$$
having the Hurst exponent’s impact in \( \hat{d}_1 \) and \( \hat{d}_2 \):

\[
\hat{d}_1 = \frac{\ln \left( \frac{S_t}{K} \right) + r\tau + \frac{1}{2}\sigma_f^2 \tau^{2H}}{\sigma_f \tau^{H}},
\]

\[
\hat{d}_2 = \hat{d}_1 - \sigma_f \tau^{H},
\]

using \( S \) as the underlying’s price at time \( t \), \( \tau \) as the option’s time to maturity, \( r \) the risk-free discount rate, \( K \) the strike price and \( \phi \) representing normal cumulative density distribution. This allows to decompose Black-Scholes implied volatility \( \sigma_{BS} \) into its fractal components (cp. Hu & Øksendal (2003)):

\[
\sigma_{BS}(\tau) = \sigma_f \tau^{H - \frac{1}{2}}.
\]

Taking the relation between Black-Scholes volatility and its fractal components permits to measure a stock’s expected momentum on a forward looking basis via the Hurst exponent \( H \). When using fractal Black-Scholes option pricing formula for the measurement of \( H \), we see it crucial to fix the option strikes to at-the-money levels in order to avoid skew and smirk effects as options’ implied volatility surfaces are typically also non-flat. However, \( K = ATM \) includes the replicated underlying payoff expectations where Put prices equal Call prices, hence delivering a stock’s term structure of expected volatility. Given one can observe \( \sigma_{BS} \) with the corresponding time to maturity \( \tau \) from option data, taking the logarithms of equation 7 yields

\[
\ln(\sigma_{BS}(\tau)) = \ln(\sigma_f) + \left( H - \frac{1}{2} \right) \ln(\tau),
\]

which allows to simply fit the ATM volatility surface over time to maturity \( \tau \) by OLS regression of equation 8,

\[
\hat{y}_i = \hat{\alpha} + \hat{\beta} \ln(\tau) + \hat{e}_i,
\]

having

\[
\sigma_f = e^{\hat{\alpha}}
\]

\[
H = \hat{\beta} + 0.5.
\]

Figure 4 displays regression fit of Equation 8 from the case of the SP500 which we had
in Figure 2 before. As shown, implementing fractal Brownian motion into standard Black-Scholes option pricing formula delivers high fit-ability of empirical data. Over the entire observation horizon, goodness of fit $R^2$ of the daily regressions from Equation 8 on the SP500 data ranged between 0.945 (1st quartile) and 0.997 (3rd quartile).

Figure 4
Logarithmic representation of Figure 2. Applying OLS regression (Equation 8) delivers a good fit of the actual ATM implied volatility surface, which allows to decompose volatility expectations into a base level ($\sigma_f$) and the trend direction ($H_t$). High fit-ability is verified by $R^2$ of 0.987 (2008-10-01) and 0.997 (2018-10-01). $H > 0.5$ indicates $\partial \sigma_{BS,t}^{ATM}(\tau)/\partial \tau > 0$ and thus increasing risk expectations over $\tau$ at $t$ (2018), vice versa holds for $H_t < 0.5$ (2008).

Consequently, we can formulate the sign and intensity of expected autocorrelation from a statistical point of view by using option’s implied Hurst exponent $H$. For an economic perspective, we introduce arbitrage free principles plus basic portfolio theory (Markowitz (1952)), from which we know that for a given point in time $t$, one can draw the capital market line between the risk free rate $r_f$ and the tangential portfolio. Basic portfolio theory implies that investors choose between the risk free rate and the tangential portfolio causing the capital market line with an intercept of $r_f$ and a slope $k$ to equal the maximal possible return-risk trade-off, thus expected return is a linear function of risk taken. Since $k$ varies over $t$, we use the notation $k_t$ instead, but still $k_t > 0, \forall t$ from basic portfolio theory has to hold. We use this concept and apply the thoughts onto the risk-return parity on forward looking contracts. Assume two futures on the market index with two different times to maturity, if the first future has higher expected return per unit risk taken than the second
one, then investors could go long into the first future and short sell the second one, until the price of the two contracts will balance in at an equilibrium risk-return parity, else arbitrage opportunities would persist. Hence, for a given point in time, the risk return trade-off from portfolio theory also has to hold for future contracts with different times to maturity. Same argumentation can be made by a replicated expected stock payoffs through Puts and Calls - higher expected risk comes with greater expected return. Following, that at time $t$ investors have one certain equilibrium expectation on the market risk-return parity, such that the expected excess market return $\mathbb{E}[\mu_{t,t+\tau}] - r_f$ for different dates in the future $t + \tau$ is an arbitrage free scalable $k_t$ of the given risk expectation, hence $k$ is constant at $t$ for different future days ahead $\tau$:

$$
\frac{\mathbb{E}[\mu_{t,t+\tau}] - r_f}{\mathbb{E}[\sigma_{t,t+\tau}]} = k_t \quad \forall \tau \quad (11)
$$

Otherwise, if Equation 11 does not hold, investors could go long into market futures with greater return per unit risk and short sell futures of the same underlying having different maturity where the risk-return parity is expected to be worse, causing arbitrage-free principles to be violated. Note that this assumption is a weaker form than the one implied by Black-Scholes, where it is assumed that all $\mathbb{E}[\sigma_{t,t+\tau}]$ are constant over all levels of $\tau$, which we know to typically not hold (cp. Fig. 4 and 7; or e.g. Li & Chen (2014)). Nonetheless, assuming that Equation 11 is valid, if for example near future $t + \tau_1$ market volatility is high but declines over days farer in the future $t + \tau_2$ (using $\tau_2 > \tau_1$), investors expect higher returns in the weeks ahead but a cooldown on the long run since $\sigma_{t,t+\tau_1} > \sigma_{t,t+\tau_2}$. Opposite would be true for an upward sloping volatility term structure. Therefore, under an implied Hurst exponent $H > 0.5$, $\sigma_{t,t+\tau}$ will be increasing with $\tau$, investors expect a positive drift in equity returns and thus positive momentum. Conversely, for $H < 0.5$ momentum believes are negative. Empirical support for this statement is given in Section 4. Under equilibrium risk-return expectation in Equation 11, we can draw different volatility believes of a stock direct into a $\mu$-$\sigma$ diagram (see Fig. 5). Given we do not know the expected market return for different dates in the future, we assume there is only one unique slope of the capital market line $k_t$ for all $t + \tau$ such that also on the level of single stocks, $H > 0.5$
means $\sigma_{t,t+\tau_2} > \sigma_{t,t+\tau_1}$ and accordingly $\mathbb{E}[\mu_{t,t+\tau_2}] > \mathbb{E}[\mu_{t,t+\tau_1}]$ and vice versa. Consequently, under arbitrage free principles where generally $k_t > 0$, we formulate our thoughts that the risk-neutral momentum expectations are defined as

**Hypothesis 1:**

$$
H > 0.5 \implies \frac{\partial \sigma_{BS}}{\partial \tau} > 0 \implies \frac{\partial \mathbb{E}[\mu]}{\partial \tau} > 0 \implies \text{positive Momentum believes}
$$

$$
H < 0.5 \implies \frac{\partial \sigma_{BS}}{\partial \tau} < 0 \implies \frac{\partial \mathbb{E}[\mu]}{\partial \tau} < 0 \implies \text{negative Momentum believes.}
$$

(12)

Empirical tests on Hypothesis 1 can be found in Section 4, where we examine the relation between SP500’s implied Hurst exponent and the corresponding realized index returns that followed.

Different to historical momentum, we find that risk-neutral momentum also has an economic argumentation, sourced in Equation 11 together with arbitrage free principles, which led us to formulate Hypothesis 2.

**Hypothesis 2:** Risk-neutral momentum is represented by the non-flatness of the expected volatility term structure, hence in absence of arbitrage, economically it can be seen as compensation for expected risk to increase/decrease in the future.

Given Equation 11, by risk-neutral ideas, positive momentum implies that also risk is expected to increase (upward sloping ATM-volatility term structure), therefore, under economic concepts stating that there is no free-lunch available in efficient markets, risk-neutral momentum is solely a compensation of this expected increase in risk (cp. Equation 12, Fig. 5). Following, from an economic perspective, one can define risk-neutral momentum directly through variation in expected risk over future days ahead, which links to increasing or decreasing expected returns. Ignoring the expected volatility term structures and not pricing this risk variation would be thus economically incomplete. Therefore, we see Hypothesis 2 to hold. Figure 5 sketches the idea of the economic argumentation within a $\mu - \sigma$ diagram on the top-left and the term structure of volatility on the right hand side, the case displayed
shows a positive drift in expected return ($H > 0.5$).

![Diagram showing expected return $E[\mu]$, $E[\sigma_{t,1}]$, $E[\sigma_{t,2}]$, and $E[\sigma_t]$ with $k_t$, $2$, and $\sigma_{BS}$ plots.](image)

**Figure 5**
Visualization of Hypothesis 2: Case of positive momentum expectations. Assume that we replicate an underlying (equity) payoff by 1:1 relation of ATM Put to Call, such that the underlying’s term structure of expected risk can be expressed via option implied volatilities. Plots show the relation between basic portfolio theory (top-left graph) and expected risk for the future $\tau$ (right graph) at time $t$. Note that $\sigma_{BS}$ is directly derived from traded option prices, hence reflects expectations per definition. Expected risk-return relation ($k_t$) varies over time, but, arbitrage free principle implies that $k_t > 0, \forall t$. This allows to observe that at $t$, the slope of the implied volatility curve $\partial \sigma_{BS} / \partial \tau$ determines whether expected returns are believed to increase or decrease for the future $\tau$. If $\partial \sigma_{BS} / \partial \tau > 0$, then absence of arbitrage requires higher expected risk to be compensated by higher expected returns (top-left graph), which leads into an upward drift of return expectations inducing positive momentum believes. Vice versa holds for $\partial \sigma_{BS} / \partial \tau < 0$. Only in the special case of $\partial \sigma_{BS} / \partial \tau = 0$, momentum expectations are zero and corresponding assumptions of CAPM, Fama & French (2015) or standard Black & Scholes (1973) hold.

Hypothesis 2 is a more general notation, which does not necessarily require to know the exact form of the underlying stochastic process. Within Hypothesis 2, $\partial \sigma_{BS} / \partial \tau > 0 \implies \partial E[\mu] / \partial \tau > 0$ is stated to hold without any connections to the Hurst exponent $H$.

**Risk-Neutral Tail Risk** Before, the focus was on the ex-ante term structure of volatilities (time dimension), now the focus is on the risk-neutral expectations of non-normality implied by the OTM surface (distributional dimension). That higher moments are indeed
of relevance was first discussed in Rubinstein (1973) and empirically observed by Kraus & Litzenberger (1976). Compared to normal distribution, we state that higher kurtosis pushes probabilities for more extreme returns. At the distribution’s third moment, we see negative skewness to cause extreme returns on the negative side and vice versa. Hence, relation to tail risk\(^5\) is positive for kurtosis and contrary for skewness. Economic relevance of tail risk can be made through two approaches. First, via Kahnemann & Tversky (1979)’s prospect theory such that under decreasing marginal utility and loss aversion, extreme returns generate less additional utility, causing investors to favor normal distributed returns over leptokurtic and skewed ones. Related finance literature comes besides others from Konno et al. (1993), Bakshi et al. (2003) or Kelly & Jiang (2014). Second economic argumentation can be achieved via Merton (1974)’s model of default. As Cremers & Weinbaum (2010) or Schneider et al. (2016) already argue, within this model, tails on the negative side of the return distribution shift probability mass below the default barrier and thus push credit risk. Therefore, for two return distributions being identical in mean and variance, there is still an economic difference whether distributions are believed to be normal or tailed, as (left) tailed distributions raise probability of default. From this argumentation we derived the wording 'tail-risk', as it is highlighted that skewness and kurtosis are clearly a (credit) risk source and not solely investor preference.

In purpose of risk-neutral measurement, we take the same starting point as before, assuming that under efficient markets, option prices as forward looking contracts correctly represent investor believes. Different to above, we now fix the time to maturity of the observed options and have a look on the B/S implied volatility shape over different levels of strikes. Also here, the theoretical line is flat as skewness and excess kurtosis are assumed to be zero under Black & Scholes (1973). However, empirical data shows that this assumption hardly holds (Fig. 6 and 7). Figure 6 displays examples of the SP500 real implied volatility surfaces for two different dates, both curves show the typical pattern that out-the-money options are priced under higher volatilities compared to in-the-money options. Reasoning why this is a common case comes from demand based option pricing (e.g. Garleanu et al.

\(^5\)‘Tail risk’ meant as the risk for large downturns, probability to realize extremely negative returns.
(2009)). As already argued via Merton (1974) model, the underlying’s price is limited by zero but unlimited on the upside, thus credit risk embeds a crucial role. Demand based option pricing now claims that investors use put options to ensure themselves against large negative returns or even bankruptcy events. This one sided limitation causes higher riskiness of OTM options, thus investors price them under higher volatility expectations.

Figure 6
Option implied Black-Scholes volatility surface of the SP500 over different degrees of moneyness at a time to maturity of one year. Under Black-Scholes assumption, this relation should be flat, empirical data show that this does not hold. From demand based option pricing we know, that skews and smirks can be linked to risk-neutral skewness and kurtosis expectations.

Finance literature brings broad evidence, that the non-flatness among B/S volatilities embeds risk-neutral information on skewness and kurtosis, Bali & Hovakimian (2009), Xing (2010), Rehmann & Vilkov (2012), An et al. (2014) or Bali, Hu & Murray (2017) just to name a few. There are different methods discussed of how to measure implied skewness and kurtosis from option prices. On index level, we take the regression approach of Backus et al. (2004). Here, the implied volatility is plotted against the option’s moneyness. Taking the approximation method from Gram-Charlier expansion (see Jarrow & Rudd (1982)), implied skewness and kurtosis can then be estimated by least-squares regression analysis of the implied volatility surface using

\[ \frac{\sigma_{BS}(d)}{\sigma_{ATM}^{BS}} = \hat{\alpha} + \hat{\beta}_0 \ast \left( \frac{d}{3!} \right) + \hat{\beta}_1 \ast \left( \frac{d^2}{4!} \right), \quad d = \frac{\ln(K/S)}{\sigma \sqrt{T}}, \]  

(13)

with \( \sigma_{BS} \) as the empirically measured implied volatility from Black-Scholes pricing, \( d \) the
standardized moneyness, $\hat{\beta}_0$ as implied skewness and $\hat{\beta}_1$ implied kurtosis. This approach requires frequently traded options on a sufficient number of moneyness levels in order to derive enough data points to run regressions reliably. In the case of the SP500 we find this requirement to hold, on the level of single stocks - especially small sized ones - the data availability is often not that complete. Therefore, on the stock market level we measure implied skewness and kurtosis according to the non-parametric approach of Bali & Hovakimian (2009) and Mixon (2010), where the implied non-normality measures are approximated by

\[
\text{skewness} = \sigma^{\Delta=0.25}_{BS,C} - \sigma^{\Delta=0.25}_{BS,P}
\]

\[
\text{kurtosis} = \sigma^{\Delta=0.25}_{BS,C} + \sigma^{\Delta=0.25}_{BS,P} - \sigma^{\Delta=0.5}_{BS,C} - \sigma^{\Delta=0.5}_{BS,P}
\]

(14)

having $\Delta$ as the sensitivity to the underlying’s price change of the Call- or Put-type option (denoted by $C$ and $P$ respectively). Following, skewness measures the skew of the implied volatility and kurtosis the degree of smile. As in the empirical evaluation on the stock market level we use those measures in purpose of roughly clustering the investment universe into quantiles, we see the non-parametric approximation to be accurate enough and the most suitable method for this task. Other empirical works also implementing this approach for option implied measurement are Cremers & Weinbaum (2010), DeMiguel et al. (2013) or Bali, Hu & Murray (2017). A comparison of risk-neutral moment estimation methods can be found in Liu & van der Heijden (2015).

3 Data

The focus of this paper is on the U.S. equity market, once on a market index level and second on the cross section of stock returns, starting in January 2005 to November 2018 including all surviving and non-surviving stocks where we had option data available. This gives us an observation universe consisting of 4376 stocks in total. All return series used are on a daily basis. Entire option and stock data are derived from Bloomberg L.P., data for the risk free rate and the CRSP market index are downloaded from the data library of Kenneth R. French\(^6\). According to the previous explanations, we compute option implied

\(^6\)http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/; accessed: 15th Nov. 2018
Hurst exponent, skewness and kurtosis as the investors’ risk-neutral expectations for the
SP500 on a daily and for single stocks on an end-of-month basis. Generally, all portfolios
formed within this paper are rebalanced on a monthly basis and value weighted according to
stocks’ one month lagged market capitalization. Option data covers time-to-maturities from
3 to 18 months and degrees of moneyness ranging 80% to 120%.

4 Empirical Patterns at Market Level

To verify Hypothesis 1 and to get insights of the relation between risk-neutral expectations
and returns, we analyze the SP500 index under its option implied volatility surface. On a
daily basis we decompose the corresponding ATM volatilities into its fractal components of
implied Hurst exponent $H$ defined by Equations 8 to 10 and related fractal volatility $\sigma_f$.
Also daily, the SP500 OTM volatilities are fitted by least squares regression from Equation
13 to measure implied skewness and kurtosis. The time series of the SP500 together with
those measures are shown and summarized in Figure 7 and Table 1.

Table 1
Summary statistics of the SP500 index and its option implied risk-neutral expectations (daily
basis). Implied Hurst exponent is mainly above 0.5 indicating that most of the time investors
expect growth in the market. Mean of implied skewness is negative, for kurtosis positive, thus
expected return distribution is mostly tailed, especially on the downside. All four risk-neutral
expectation measures turn out to be on average not to far away from realized (physical) patterns.
Details on the distribution of implied expectations versus rolling physically realizations are attached
at Appendix B. Return displayed in percent monthly.

<table>
<thead>
<tr>
<th></th>
<th>return</th>
<th>Hurst</th>
<th>frac.Vol.</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Min.</td>
<td>-9.470</td>
<td>0.272</td>
<td>0.024</td>
<td>-2.054</td>
<td>-0.283</td>
</tr>
<tr>
<td>Median</td>
<td>0.065</td>
<td>0.601</td>
<td>0.092</td>
<td>-0.831</td>
<td>0.282</td>
</tr>
<tr>
<td>Mean</td>
<td>0.024</td>
<td>0.590</td>
<td>0.139</td>
<td>-0.903</td>
<td>0.315</td>
</tr>
<tr>
<td>(physical)</td>
<td>(-0.378)</td>
<td>(0.187)</td>
<td>(-0.378)</td>
<td>(0.120)</td>
<td></td>
</tr>
<tr>
<td>Max.</td>
<td>10.957</td>
<td>0.787</td>
<td>1.445</td>
<td>0.312</td>
<td>0.846</td>
</tr>
<tr>
<td>sd</td>
<td>1.176</td>
<td>0.076</td>
<td>0.140</td>
<td>0.370</td>
<td>0.171</td>
</tr>
</tbody>
</table>

As measured on SP500 daily returns over the entire observation horizon. Physical Hurst is computed
by simple R/S analysis.
Figure 7
Price development of the SP500 index and the corresponding momentum (implied Hurst) and tail-risk believes (implied skewness and kurtosis). A Hurst exponent below 0.5 indicates that investors expect negative momentum on the market, above 0.5 determines growth believes. The implied Hurst indicates nicely wild and mild times, during the financial crisis, the Hurst exponent turns largely below 0.5, this means that investors expected a decline in returns, which came close to future realization. In normal times, the Hurst exponent indicates growth, only during sideways or short downward trends, this coefficient turned negative, making the implied Hurst exponent a qualified candidate for risk-neutral momentum measurement. Both, implied skewness and kurtosis are seen as drivers of tail risk. As the relation between tail risk and skewness is negative but positive for kurtosis, it is not surprisingly that the stochastic movement is very similar. Realized correlation between implied skewness and implied kurtosis is -0.98. T-tests on the Hurst exponent and the higher moments confirm that all three measures are highly significantly different from zero, supporting our claim that common assumptions on momentum and normally distributed returns do not hold empirically.
Going into details to understand the relation between implied Hurst and SP500 returns, we directly compare daily returns having $H_t > 0.5$ to those of $H_t < 0.5$ within a two-sample t-test. This shows that $r_{t|H_t>0.5}$ are significantly greater than $r_{t|H_t<0.5}$ at an t-statistic of $t = 3.85$ (p-value = 6.7e-05), giving support for Hypothesis 1. Related mean returns are 20.7% ($r_{t|H_t>0.5}$), -9.8% ($r_{t|H_t<0.5}$) and 6.0% for the SP500 at standard deviations of 12.9%, 39.9% and 18.7% per annum respectively. Since Figure 7 indicates that the implied Hurst exponent is mainly negative during stagnating and downward market phases, the t-test results are not surprisingly. Consequently we see $H$ to also be a measure for nervousness of the market.

As with the implied Hurst exponent we aim to measure momentum expectations, we analyze the relation between implied Hurst and future realized returns in a predictive regression framework defined by 

$$r^{SP500}_{t\rightarrow t+\tau} = \hat{\alpha} + \hat{\beta} \ast (H_t - 0.5),$$

(15)

where we test under three different time horizons $\tau$: 1 week, 1 month and 1 year. Given lower fluctuation in the one-year forward return, the predictive regression for $\tau = 1$ year is made on a monthly basis to avoid over-fitting bias, the other two regressions are on daily basis as they face shorter forward horizons. $H_t$ is subtracted by 0.5 to allow a direct interpretation of $\hat{\beta}$, as 0.5 is the critical threshold of $H$ indicating momentum direction. Table 2 displays the corresponding regression outputs.

---

8Recap Hypothesis 1, linking $H > 0.5$ to positive and $H < 0.5$ to negative ex-ante momentum expectations.
Predictive regression (Equation 15) of SP500’s implied Hurst exponent with the related forward return of 1 week, 1 month and 1 year. For all three time horizon we see significant and positive relation between expected momentum (implied Hurst) and future performance, thus supports Hypothesis 1.

<table>
<thead>
<tr>
<th></th>
<th>$r_{t+\tau}^{SP500}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\tau = 1$ week</td>
</tr>
<tr>
<td>Implied Hurst</td>
<td>0.019***</td>
</tr>
<tr>
<td></td>
<td>(0.005)</td>
</tr>
<tr>
<td>Intercept</td>
<td>-0.001</td>
</tr>
<tr>
<td></td>
<td>(0.001)</td>
</tr>
<tr>
<td>Observations</td>
<td>3,471</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.004</td>
</tr>
<tr>
<td>F Statistic</td>
<td>13.760***</td>
</tr>
</tbody>
</table>

Note: *p<0.1; **p<0.05; ***p<0.01

We observe that $H_t$ is indeed followed by positive returns, making it a valid candidate for risk-neutral momentum approximation and confirmation of Hypothesis 1. Given this significant predictive potential, we are interested whether investors can use this insight to time their investments in the SP500. Therefore, we directly compare SP500 returns under positive momentum believes with those of negative expected trends. Doing so, we split our SP500 data set into two sub samples. In the first sample, we replace the daily SP500 returns where $H_t < 0.5$ with 0, meaning that an investor would be invested in the index only if momentum believes are positive ($\mathbb{E}[M_t] > 0$). The second sample represents the opposite, an investment in the SP500 only for days where expected momentum is negative ($\mathbb{E}[M_t] > 0$):

$$
\begin{align*}
    r_{t}^{\mathbb{E}[M_t] > 0} &= \begin{cases} 
        r_{t}^{SP500}, & \text{if } H_t > 0.5 \\
        0, & \text{otherwise}
    \end{cases} \\
    r_{t}^{\mathbb{E}[M_t] < 0} &= \begin{cases} 
        r_{t}^{SP500}, & \text{if } H_t < 0.5 \\
        0, & \text{otherwise}
    \end{cases}
\end{align*}
$$

This separation allows us to compare the two cases directly, having same length of observations. From Table 2 we see, that $H_t$’s predictive ability on future returns is greater
the longer the time window, however, with more days ahead the momentum effect is more likely to be mixed by other effects. Accordingly, for strategy back-testing we introduce a minimum holding period of one week. Under this setting, the relation between risk-neutral momentum expectation $\mathbb{E}[M]$ at time $t$ with the following future returns is tested by OLS regression on 1 week (5 trading days) forward returns of the two samples ($\mathbb{E}[M_t] > 0$ and $\mathbb{E}[M_t] < 0$) against the SP500:

$$
\mathbb{E}[M_t] \leq 0 \\
\mathbb{E}[M_t] > 0
$$

(17)

Results can be found in Table 3. Same strategies as for momentum (Equation 17) are analyzed for ex-ante skewness and kurtosis but with the difference, that for those tail risk measures we control the investment quote not by whether they are greater or less than zero, but if the implied measure is above or below its time cumulated average. With this methodology we want to determine how unnatural high expectations on the distribution’s tails relate to index performance, corresponding results are displayed in Table 3. From this analysis we derive the same conclusion as before, positive momentum returns produce significant positive intercepts, while negative momentum expectations are followed by under-performance, hence again gives confirmation of Hypothesis 1. The two tail risk cases are in line with each other, meaning that for both measures, always the strategy where tail risk is expected to be unusual worse generates significant negative alphas, which can be linked to the distress puzzle (cp. Campbell et al. (2008), Schneider et al. (2016)) from an perspective of investment timing. Drawback coming from the short holding horizon of one week is that it induces more trades, for the momentum strategy it were in total 122 buying and selling trades, for skewness and kurtosis 234 and 283 respectively. Following, significance of alphas may disappear after accounting for transaction costs. Nonetheless, empirical patterns found give details and hints for momentum and tail risk timing anomalies, related style anomalies are tested on a cross sectional level in the following of the reading.
Table 3
SP500 timing strategies. Comparing the two momentum ($M_t$) cases: $r^E_{t|M_t>0}$ investors are only invested in the SP500 on days where momentum believes are positive vs. the opposite case ($r^E_{t|M_t<0}$).

From OLS regression we see, that days with positive momentum believes are followed by returns with significant high alphas and $H_t < 0.5$ days will lead into under-performance. Therefore, we find support for Hypothesis 1 and link $H_t > 0.5$ to positive momentum believes and $H_t < 0.5$ to negative trend expectations. Also in terms of tail risk we find time-dependent anomalies such that times where tail risk falls below its average are followed by worse performance compared to its opposites. Tail risk is measured once (i) upon implied skewness ($iS$), where more negative skewness is linked to higher tail risk and (ii) kurtosis ($iK$), where a higher measure induces more likelihood for extremes. Analysis made on daily data, returns are scaled to percent monthly.

<table>
<thead>
<tr>
<th></th>
<th>momentum</th>
<th>skewness</th>
<th>kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E[M_t]&gt;0$</td>
<td>$E[M_t]&lt;0$</td>
<td>$iS_t&gt;avg$</td>
</tr>
<tr>
<td>$r^{SP500}_{t\rightarrow t+5}$</td>
<td>0.517***</td>
<td>0.483***</td>
<td>0.457***</td>
</tr>
<tr>
<td>(0.008)</td>
<td>(0.008)</td>
<td>(0.008)</td>
<td>(0.008)</td>
</tr>
<tr>
<td>intercept</td>
<td>0.392***</td>
<td>-0.392***</td>
<td>0.220***</td>
</tr>
<tr>
<td>(0.0002)</td>
<td>(0.0002)</td>
<td>(0.0002)</td>
<td>(0.0002)</td>
</tr>
<tr>
<td>mean</td>
<td>0.650</td>
<td>-0.142</td>
<td>0.575</td>
</tr>
<tr>
<td>sigma</td>
<td>3.435</td>
<td>3.320</td>
<td>3.030</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.517</td>
<td>0.484</td>
<td>0.456</td>
</tr>
</tbody>
</table>

Note:  
*p<0.1; **p<0.05; ***p<0.01

Predicting Market Volatility  How about the predictability from the expected volatility surface on future realized volatility? As we find high significance within empirical patterns of implied expectations in relation to returns, we suggest that option contained information also qualifies to be a valid predictor of future realized volatility. To examine this claim, we make out-of-sample analysis of volatility prediction based on a simple machine learning algorithm. The algorithm’s underlying model is standard linear regression, realized future volatility over $\tau$ at the time $t$ is defined as $\hat{\sigma}(\tau) = \sigma_{t\rightarrow t+\tau}$ and $\hat{\sigma}(\tau)$ denotes the corresponding model estimate. Training of the model and predictions are both made daily. The train set is constructed time expanding covering $t_0 \rightarrow t - \tau$, estimation is then made by applying the trained model on the separate test set $t \rightarrow t + \tau$. As explanatory variables we chose fractal volatility $\sigma_{f,t}$, implied Hurst exponent $H_t$ subtracted by 0.5 and implied skewness $iS_t$. Observing high correlations between skewness and kurtosis $iK_t$ (see Table 4 Panel A),
the later variable is not included in the predictive regression. To ensure robustness of this approach, four different prediction horizons were chosen: \( \tau = 1 \) day, 1 week, 2 weeks and 1 month. Table 4 shows details on prediction performance of the algorithm. Indeed, the predictive model achieves quiet high correlations - ranging from 0.534 to 0.796 - between estimated values and future realized ones, this can also be interpreted from goodness of fit via \( R^2 \). Only next-day-volatility prediction seems to be less reliable (\( R^2 = 0.285 \)). Analyzing the prediction errors \( |\bar{\sigma} - \hat{\sigma}| \) in more detail, mean and median are close to zero illustrating quality of this approach, also maximum errors remain in an acceptable range. As conclusion, when it comes to the task of predicting volatilities, we see information regarding investor expectations contained within the implied volatility surface to be a potent candidate, which further supports the relevance of our empirical findings.

Table 4
Volatility estimation of the SP500: Predictive performance of machine learning algorithm, building on linear regression using information contained in option’s implied volatility surface. Model re-training and out-of-sample prediction made daily. Train-test set split is formed on an time expanding basis without overlapping the two sets. Overall predictability turns out to be quiet high for all horizons, which can be seen from correlations between \( \bar{\sigma}, \hat{\sigma} \) and from \( R^2 \). Prediction error \( |\bar{\sigma} - \hat{\sigma}| \) (presented in percentage points) is on average small and within an acceptable range. Thus, we find option implied volatility surface to be of high value when estimating SP500’s future realized volatility. The good fit of this approach can also be discovered visually from the Plot presented in Appendix C.

| \( \tau \)          | \( \text{cor}(\bar{\sigma}, \hat{\sigma}) \) | \( R^2 \) | \( |\bar{\sigma} - \hat{\sigma}| \) |
|------------------|------------------|--------|------------------|
| \( 1 \) day      | 0.534            | 0.285  | 0.000            |
| \( 1 \) week     | 0.803            | 0.644  | 0.001            |
| \( 2 \) weeks    | 0.830            | 0.689  | 0.000            |
| \( 1 \) month    | 0.796            | 0.633  | 0.000            |

\[ \hat{\sigma}_t = \hat{\alpha} + \hat{\beta}_1 (H_{t-1} - 0.5) + \hat{\beta}_2 \sigma_{f,t-1} + \hat{\beta}_3 iS_{t-1} + \hat{\epsilon} \]

\( \text{Note that we exclude } t - \tau \text{ data points from the train set in order to avoid overlaps between train and test sets. } t_0 = \text{Jan. 5th 2005}, t \text{ for the first model train is 504 days, meaning that the initial training set covers 2 years of observations. One day lag of explanatory variables considered when predicting on test set. Hence, overall between 2950 to 2970 regressions were run for every prediction horizon } \tau. \)

\( \text{Since 0.5 is the crucial threshold of the Hurst exponent indicating whether autocorrelation is positive or negative, subtracting 0.5 reduces the predictor’s intercept.} \)
5 Empirical Patterns at the Cross Section

Decomposing SP500’s OTM and ATM implied volatility surface gives insights into timing-related anomalies sourced in ex-ante momentum and tail risk. On the cross section of U.S. equity, we go into details on those anomalies from a stock characteristic perspective. For this purpose, risk-neutral measures are computed on a monthly basis for each of the 4376 stocks in our dataset. Then, stocks are ex-ante sorted according to (i) risk-neutral momentum and (ii) expected tail risk to build comparable portfolios by value weighting the followed realized daily returns on stock’s one-month-lagged market capitalization.

Momentum Anomaly Forming ten portfolios on stock’s implied Hurst exponent shows a consistent pattern of how momentum expectation positively predict realized performance, details are in Table 5, having portfolio 1 = negative momentum expectations up to portfolio 10 with greatest believes and a long short portfolio (High-Low) which euquals the daily returns of 10 minus 1. Finance literature broadly finds that historical momentum positively relates to portfolio performance (e.g. Jegadeesh & Titman (1993), Carhart (2012)) - we find same patterns building on ex-ante momentum instead. This statement holds under both absolute and risk adjusted performance, where especially on the negative momentum side realized alphas are of great significance. As the High-Low portfolio shows significant alphas even under Fama & French (1993) three factor model, we believe that ex-ante momentum covers information content on equity returns which cannot be captured by common asset pricing factors. Additionally, realized CAPM alpha standardized by beta ($\alpha/\beta$) is displayed to show that this anomaly is consistent even after correcting for the beta anomaly (related literature can be found in e.g. Frazzini & Pedersen (2014), Bali, Brown, Murray & Tang (2017)).

Following, regression analysis from Table 5 uncovers ex-ante momentum to positively predict future absolute and relative performance. Results are cross validated by tests applying Carhart (2012) four factor and Fama & French (2015) five factor pricing models with the conclusion, that the effect found is also robust under those frameworks. Respective alphas

---

9 Portfolio ex-ante $H$ is estimated monthly as the value weighted average of stock ex-ante $H$, displayed values are mean portfolio $H$ over observation horizon. Physical estimation methods are based on R/S analysis with different approaches, details and comparison can be found in Weron (2002).
Table 5
Ten value weighted portfolios sorted on ex-ante momentum expectations, 1=low momentum up to 10=high momentum. The greater momentum expectations the higher are mean returns and realized alphas, thus we see existence of an ex-ante momentum anomaly. The High-Low portfolio takes the long short positions of portfolio 10 minus 1. As High-Low portfolio is significant with a p-value below 1% we see the statement of momentum anomaly to hold. Effect still remains persistent when testing under Fama-French three factor model (FF3). Predictive ability of ex-ante momentum on future realized autocorrelation is confirmed by analyzing portfolios’ physically realized $H$ exponents, having increasing tendency for realized over ex-ante $H$ under all four tested physical estimation methods of the Hurst exponent. This table also shows the problem of physical approximation of the Hurst exponent, as the output differs among method applied.

<table>
<thead>
<tr>
<th></th>
<th>negative Momentum</th>
<th>positive Momentum</th>
<th>High-Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>alpha</td>
<td>-1.03</td>
<td>-0.61</td>
<td>1.07</td>
</tr>
<tr>
<td></td>
<td>(***</td>
<td>(**</td>
<td></td>
</tr>
<tr>
<td>alpha (FF3)</td>
<td>-0.97</td>
<td>-0.59</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>(***</td>
<td>(**</td>
<td></td>
</tr>
<tr>
<td>beta</td>
<td>1.28</td>
<td>1.25</td>
<td>-0.43</td>
</tr>
<tr>
<td></td>
<td>(***</td>
<td>(**</td>
<td></td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.80</td>
<td>0.90</td>
<td>0.24</td>
</tr>
<tr>
<td>$\alpha/\beta$</td>
<td>-0.10</td>
<td>-0.06</td>
<td>-0.30</td>
</tr>
<tr>
<td>mean</td>
<td>-0.32</td>
<td>0.11</td>
<td>0.91</td>
</tr>
<tr>
<td>volatility</td>
<td>8.17</td>
<td>7.50</td>
<td>4.99</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>-0.05</td>
<td>0.00</td>
<td>0.16</td>
</tr>
</tbody>
</table>

$Hurst exponents$

<table>
<thead>
<tr>
<th></th>
<th>ex-ante $H$</th>
<th>.347</th>
<th>.445</th>
<th>.470</th>
<th>.485</th>
<th>.498</th>
<th>.507</th>
<th>.517</th>
<th>.532</th>
<th>.553</th>
<th>.604</th>
</tr>
</thead>
<tbody>
<tr>
<td>physical</td>
<td>simple R/S</td>
<td>.505</td>
<td>.517</td>
<td>.505</td>
<td>.514</td>
<td>.504</td>
<td>.521</td>
<td>.513</td>
<td>.525</td>
<td>.534</td>
<td>.521</td>
</tr>
<tr>
<td></td>
<td>corr. R/S</td>
<td>.504</td>
<td>.531</td>
<td>.521</td>
<td>.517</td>
<td>.512</td>
<td>.523</td>
<td>.542</td>
<td>.539</td>
<td>.557</td>
<td>.546</td>
</tr>
<tr>
<td></td>
<td>empirical</td>
<td>.478</td>
<td>.512</td>
<td>.507</td>
<td>.486</td>
<td>.491</td>
<td>.498</td>
<td>.536</td>
<td>.504</td>
<td>.531</td>
<td>.537</td>
</tr>
<tr>
<td></td>
<td>corr. emp.</td>
<td>.448</td>
<td>.481</td>
<td>.475</td>
<td>.455</td>
<td>.460</td>
<td>.467</td>
<td>.504</td>
<td>.473</td>
<td>.498</td>
<td>.504</td>
</tr>
</tbody>
</table>

*Note:* $^*$p<0.1; $^{**}$p<0.05; $^{***}$p<0.01
of the High-Low portfolio are 0.94*** and 0.91*** at R² of 0.41 and 0.42. Consequently, not only High-Low's intercepts are of high significance, also R² of the long short portfolio are distinctively small in all models tested, which hardens the presumption that risk-neutral momentum is scarcely explained by common asset pricing models.

**Tail-Risk Anomaly** Also for OTM options, implied information content is computed on a monthly single stock basis, ex-ante skewness and kurtosis are approximated by Equation 14. The portfolio forming here is based on double sorts¹⁰, as we link both, the distributions 3rd and 4th moment to be a source of tail risk. But before going into double sorts, we run single sorts each on skewness/kurtosis only, displayed in Table 6 Panel A-B. From this categorization we see, that ex-ante skewness positively predicts realized performance - relatively as measured by alpha and absolutely by mean return - and ex-ante kurtosis embeds a negative relation, hence both measures indicate that greater tail risk causes worse future returns. As tail risk’s economic impact can be directly linked to credit risk via Merton (1974) model - since excess kurtosis and negative skewness push probabilities of default - the empirical patterns found confirm the existence of a ’distress puzzle’¹¹. Taking the two insights, we monthly cluster the equity universe into 33% quantiles of ex-ante skewness and independently into same size categories of kurtosis, which creates a 3x3 matrix of stock classification from low to high skewness/kurtosis. Stocks within every matrix entry form a (value weighted) portfolio, the one of high kurtosis and low (negative) skewness are assigned to high tail risk, the very opposite one to low tail risk, portfolios in between are built by the averages of the matrix’s diagonals. Table 6 Panel C shows the corresponding outputs of the double sorts, from which we observe an even stronger pronunciation of the distress puzzle and a greater significance of the High-Low portfolio. As conclusion, accounting for both skewness and kurtosis allows a better isolation of the tail risk effect than taking only one of the higher moments measures (like e.g. Schneider et al. (2016)). Given high significance of High-Low portfolio’s intercept, we see ex-ante tail risk to also have ability in describing equity returns beyond common asset pricing. With falling α/β over portfolio number, the

---

¹⁰By double sorts it is meant that at rebalancing date, the cross section is sorted on two different stock characteristics.

¹¹Distress puzzle is understood as the empirical observation that firms facing high credit risk under-perform. Finance literature already enjoys a vivid discussion on this topic, e.g. Campbell et al. (2008).
tail risk anomaly is suggested to remain consistent even after correcting for beta related mispricing (beta anomaly). Significance of this High-Low portfolio persists when testing under Fama & French (1993) regression model. Results are again tested for robustness under Carhart (2012) four- ('CAR4') and Fama & French (2015) five ('FF5') factor model. Also here, the common frameworks show weak ability in explaining the risk-neutral characteristic with small $R^2$ of 0.04 (CAPM), 0.16 (CAR4) and 0.15 (FF5). Respective realized alphas for the later two models are -0.81*** and -0.77***. From that follows, that ex-ante tail risk negatively predicts absolute and relative stock performance.
Table 6
Portfolios sorted on ex-ante distributional characteristics. Panel A displays portfolios built on risk-neutral skewness (1 = negative to 10 = positive), the empirical pattern found shows that skewness positively predicts risk-adjusted performance as alpha is increasing over levels of skewness. Panel B demonstrates the results from kurtosis sorts, here the relation between current kurtosis and followed returns is negative both absolutely as mean and relatively by alpha. Combining these two moments at Panel C, we measure tail risk by double sorts on 33% quantiles. Double sorting allows an better observation of the desired tail risk effect, as alpha from the high minus low portfolio is of greater significance and more distinctive from zero compared to single sorts only at Panel A and B. Presented values are computed from daily returns, scaled to monthly units.

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>negative skewness</td>
<td>positive skewness</td>
<td>low tail-risk</td>
</tr>
<tr>
<td>1   2   3   4   5   6   7   8   9   10</td>
<td>1   2   3   4   5   6   7   8   9   10</td>
<td>1   2   3</td>
</tr>
<tr>
<td>alpha</td>
<td>-0.36</td>
<td>-0.14</td>
</tr>
<tr>
<td>beta</td>
<td>0.74</td>
<td>0.80</td>
</tr>
<tr>
<td>(*   (<em><strong>) (</strong></em>)) (<em><strong>)) (</strong>)) (<strong>)) (</strong>)) (</em><strong>)) (</strong><em>)) (</em><strong>)) (</strong>*))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean</td>
<td>0.33</td>
<td>0.60</td>
</tr>
<tr>
<td>sigma</td>
<td>5.19</td>
<td>5.22</td>
</tr>
<tr>
<td>high kurtosis</td>
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<td>0.16</td>
</tr>
<tr>
<td>beta</td>
<td>0.73</td>
<td>0.72</td>
</tr>
<tr>
<td>(<strong>)) (</strong>)) (<strong>)) (</strong>)) (<strong>)) (</strong>)) (<strong>)) (</strong>)) (?)) (<strong>)) (</strong>))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean</td>
<td>0.84</td>
<td>0.84</td>
</tr>
<tr>
<td>sigma</td>
<td>4.59</td>
<td>4.45</td>
</tr>
<tr>
<td>high tail-risk</td>
<td>alpha</td>
<td>0.40</td>
</tr>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>alpha (FF3)</td>
<td>0.37</td>
<td>0.12</td>
</tr>
<tr>
<td>(<strong>)) (</strong><em>)) (</em><strong>)) (</strong><em>)) (</em><strong>)) (</strong><em>)) (</em><strong>)) (</strong><em>)) (</em><strong>)) (</strong><em>)) (</em>**))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>beta</td>
<td>0.65</td>
<td>0.70</td>
</tr>
<tr>
<td>(<strong>)) (</strong><em>)) (</em><strong>)) (</strong><em>)) (</em><strong>)) (</strong><em>)) (</em><strong>)) (</strong><em>)) (</em><strong>)) (</strong><em>)) (</em>**))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>R²</td>
<td>0.77</td>
<td>0.83</td>
</tr>
<tr>
<td>alpha/β</td>
<td>0.62</td>
<td>0.22</td>
</tr>
<tr>
<td>mean</td>
<td>1.01</td>
<td>0.82</td>
</tr>
<tr>
<td>sigma</td>
<td>4.01</td>
<td>4.20</td>
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<tr>
<td>Sharpe ratio</td>
<td>0.23</td>
<td>0.17</td>
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</table>

Note: *p<0.1; **p<0.05; ***p<0.01
So to say, empirical patterns found within the cross section match the conclusion we made at a market level - namely, exposure to ex-ante tail risk causes worse performance.

**Further Robustness** To support our findings, we investigate the high-minus-low portfolios in more detail. For ex-ante tail risk, we take the already formed high-low portfolio from Table 6 and name it the 'TMN' risk factor. Ex-ante momentum risk 'MR' is constructed from high minus low quintiles of momentum sorts. Both risk factors are subtracted by the risk-free rate to represent excess returns. Validity of the two risk-neutral factors is tested by OLS regression under common asset pricing models (see Table 7). Both factors turn out to be of high significance and different from zero in their intercepts. Further, the exposure to all other common risk factors is very low ranging within ±0.36. Moreover, all of the eight regressions run realize minor goodness of fit with $R^2$ below 16%. These three pattern together allow us to suggest that the two risk-neutral factors have descriptive potential on stock returns which is not covered by the common models presented, where factors are built on backward looking data. Additional robustness of risk-neutral factors’ potential at asset pricing is given by GRS tests (Gibbons et al. (1989)) on four style sorted portfolios. The focus here is set on four pricing frameworks: (i) basic CAPM, (ii) Fama & French (1993) three factor model (FF3) and (iii) CAPM extended by the two risk-neutral factors TMN and MR, named 'CTM'. Table 7 Panel B gives insights on the GRS analysis. Generally, we find that under this short comparison, CTM evolves best at the GRS test. Given not only better GRS scores, also p-values are of less significance allowing us to state that MR and TMN are of quality in describing unrecognized risk sources.

Consistency of MR and TMN is also given over evolution of time. Figure 8 displays yearly absolute and risk-adjusted (Fama & French (1993) alpha) performance of the two risk-neutral factors. Matching the picture found from regression analysis before, MR turns out to be mainly positive for every year at absolute and risk-adjusted measures, while returns of TMN are throughly negative. Respectively, we see robustness in expectational built risk sources also when applying inter-temporal analysis.
Table 7
Panel A: Risk-neutral factors for momentum (MR) and tail risk (TMN) tested by OLS regression under common asset pricing frameworks of CAPM, Fama & French (1993) three factor (FF3), Carhart (2012) four factor (CAR4) and Fama & French (2015) five factor (FF5) model. Three things give MR and TMN high potential for equity pricing: First, both MR and TMN realize low and significant exposure to all other risk factors in every of the eight tested settings, ranging from -0.36 to 0.34. Second, all intercepts are of great significance and largely different from zero. Third, all asset pricing models deliver goodness of fit ($R^2$) at very low levels. These three patterns give robustness for the two risk-neutral factors to contain information which is not captured by common models. Alpha as risk-adjusted return monthly. Panel B: Comparing two classical models with CAPM extended by TMN and MR (CTM) under GRS test on different portfolio sorts. Within all four sorts, CTM achieves a better GRS-score at slightly greater p-values in contrast to its opponents.

### Panel A: Regression analysis of factors.

<table>
<thead>
<tr>
<th></th>
<th>MR</th>
<th></th>
<th></th>
<th>TMN</th>
<th></th>
<th></th>
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<tr>
<td></td>
<td>CAPM</td>
<td>FF3</td>
<td>CAR4</td>
<td>FF5</td>
<td>CAPM</td>
<td>FF3</td>
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<tr>
<td>Mkt.RF</td>
<td>-0.044***</td>
<td>0.014*</td>
<td>0.016**</td>
<td>0.021***</td>
<td>0.110***</td>
<td>0.027***</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.007)</td>
<td>(0.008)</td>
<td>(0.008)</td>
<td>(0.010)</td>
<td>(0.010)</td>
</tr>
<tr>
<td>SMB</td>
<td>-0.361***</td>
<td>-0.361***</td>
<td>-0.357***</td>
<td>0.339***</td>
<td>0.340***</td>
<td>0.305***</td>
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<tr>
<td></td>
<td>(0.015)</td>
<td>(0.015)</td>
<td>(0.015)</td>
<td>(0.020)</td>
<td>(0.020)</td>
<td>(0.021)</td>
</tr>
<tr>
<td>HML</td>
<td>-0.060***</td>
<td>-0.051***</td>
<td>-0.063***</td>
<td>0.208***</td>
<td>0.124***</td>
<td>0.142***</td>
</tr>
<tr>
<td></td>
<td>(0.013)</td>
<td>(0.015)</td>
<td>(0.015)</td>
<td>(0.018)</td>
<td>(0.021)</td>
<td>(0.020)</td>
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<tr>
<td>UMD</td>
<td>0.012</td>
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<td>-0.113***</td>
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<td>(0.014)</td>
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<td>(0.010)</td>
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<td>RMW</td>
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<td>-0.244***</td>
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<td>0.045*</td>
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<td>(0.026)</td>
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<tr>
<td>CMA</td>
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<td></td>
<td></td>
<td>0.151***</td>
<td></td>
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<tr>
<td></td>
<td>0.064**</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(0.029)</td>
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<td></td>
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<tr>
<td>alpha</td>
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<td>0.419***</td>
<td>0.416***</td>
<td>0.403***</td>
<td>-0.864***</td>
<td>-0.830***</td>
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<td></td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0001)</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.010</td>
<td>0.165</td>
<td>0.167</td>
<td>0.036</td>
<td>0.139</td>
<td>0.155</td>
</tr>
<tr>
<td>F Stat.</td>
<td>35.744***</td>
<td>223.670***</td>
<td>168.154***</td>
<td>136.105***</td>
<td>125.652***</td>
<td>182.420***</td>
</tr>
</tbody>
</table>

Note: *p<0.1; **p<0.05; ***p<0.01

### Panel B: Comparison by GRS test.

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</tr>
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<td>p-value</td>
<td>p-value</td>
<td>GRS</td>
<td>p-value</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>4.86</td>
<td>5.5e-07</td>
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<td></td>
<td></td>
<td></td>
<td>5.08</td>
<td>1.3e-05</td>
</tr>
<tr>
<td>FF3</td>
<td>6.83</td>
<td>5.6e-23</td>
<td>7.71</td>
<td>5.8e-27</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>4.96</td>
<td>3.6e-07</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4.50</td>
<td>2.5e-06</td>
</tr>
<tr>
<td>CTM</td>
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<td>7.24</td>
<td>8.1e-25</td>
</tr>
<tr>
<td></td>
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<td>3.92</td>
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<td></td>
<td></td>
<td>3.75</td>
<td>4.9e-05</td>
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</table>

31
Figure 8
Yearly mean returns (bars) and alphas (points, crosses) of high-minus-low portfolios on expected (i) tail risk (TMN) and (ii) expected momentum (MR). Both, the tail-risk and the momentum anomalies show consistency over the observed time period as TMN is throughout negative and MR mainly positive under both absolute and relative measures (mean return and alpha). Alphas are measured by Fama & French (1993) regression on a yearly setting, thus giving time conditional intercepts. This pattern confirms that ex-ante tail-risk and momentum effects are robust over time.

6 Conclusion
This work puts its focus on model assumptions that are frequently made at asset pricing, namely normally distributed and un-autocorrelated stock returns. We follow an risk-neutral approach using option implied volatilities such that actually traded investor expectations are measured in advance. Applying arbitrage free principles and fractal stochastics, we connect option implied Hurst exponent as risk-neutral autocorrelation proxy (Hu & Øksendal (2003)) to ex-ante momentum, show how it can be quantified and hypothesize that our formulation is sourced in an economic reasoning. From Merton (1974) model of credit risk we argue, that

skewness negatively and kurtosis positively relates to probability of default, hence we refer to distribution’s 3rd and 4th moments as tail risk.

Building on our theoretical part, our empirical work is made on two levels. First, on a market level where we examine how the ex-ante measures relate to forward returns by timely decomposition of the SP500 index. From regression analysis we find, that days with above average high ex-ante tail risk are followed by days which significantly under-perform (-3.6%*** alpha\textsuperscript{13} p.a.). Contrary, positive ex-ante momentum in the SP500 is followed by greater risk-adjusted returns (4.7%*** alpha p.a.). In terms of completeness, we make a short digression to out-of-sample prediction of SP500 future realized volatility from risk-neutral components implied by option data. Applying machine learning, we conclude that ex-ante momentum and tail risk are valuable candidates in this task with sounding predict-ability, e.g. R\textsuperscript{2} of 64.4% for the SP500 one-week-ahead volatility prediction.

On a cross sectional level, we accumulate single U.S. stocks upon the risk-neutral characteristics to build portfolios. Analyzing the daily return series of the respective monthly rebalanced and value weighted portfolios allows a very similar interpretation as we already found on the market level. Ex-ante momentum comes with greater (absolute and risk-adjusted) returns while opposite holds for tail risk. The high minus low portfolio being long in high- and short in low ex-ante momentum (MR) stocks generates economically large alpha (4.8%*** p.a.\textsuperscript{14}) at weak risk factor exposures (±0.36). The respective high-minus-low ex-ante tail risk portfolio (TMN) realizes severe under-performance of -9.2%*** alpha p.a., also keeping factor loadings minor (±0.3). Observed performances further show consistency within yearly return decomposition and embed potential for equity pricing purposes from GRS-tests (Gibbons et al. (1989)).

\textsuperscript{13}Under Fama & French (1993) three factor model.
\textsuperscript{14}Under Fama & French (2015) five factor model, to show that effects hold under different frameworks.
References


**URL**: https://doi.org/10.1111/j.1540-6261.1997.tb03808.x


**URL**: https://doi.org/10.1111/1467-9965.00018


**URL**: https://doi.org/10.1111/j.1540-6261.1970.tb00518.x


**URL**: https://www.sciencedirect.com/science/article/pii/0304405X93900235


**URL**: https://www.sciencedirect.com/science/article/pii/S0304405X14002323


**URL**: http://www.ssrn.com/abstract=2793927

**URL:** http://linkinghub.elsevier.com/retrieve/pii/S0304405X13002675


**URL:** http://www.jstor.org/stable/1913625


**URL:** https://doi.org/10.1142/S0219025703001110


**URL:** https://doi.org/10.1080/02626665609493644


**URL:** https://www.sciencedirect.com/science/article/pii/0304405X82900071


**URL:** http://www.jstor.org/stable/2328882
Kahnemann, D. & Tversky, A. (1979), ‘Prospect theory: An analysis of decision under

27(10), p.2841—2871.


Kraus, A. & Litzenberger, R. H. (1976), ‘Skewness preference and the valuation of risk

Latané, H. A. & Rendleman, R. J. (1976), ‘STANDARD DEVIATIONS OF STOCK
PRICE RATIOS IMPLIED IN OPTION PRICES’, *The Journal of Finance*
31(2), 369–381.


Variance Term Structure Model’.

Lintner, J. (1965), ‘The Valuation of Risk Assets and the Selection of Risky Investments in

*SSRN Electronic Journal (accessed June 19, 2018)*.

URL: [http://dx.doi.org/10.2139/ssrn.2641559](http://dx.doi.org/10.2139/ssrn.2641559)

59(5), 1279–1313.


**URL:** http://www.jstor.org/stable/2978814?origin=crossref


**URL:** http://arxiv.org/abs/0901.0762


**URL:** https://dx.doi.org/10.2139/ssrn.1301648


**URL:** https://dx.doi.org/10.2139/ssrn.2858933


**URL:** http://linkinghub.elsevier.com/retrieve/pii/S0378437102009615
APPENDIX

A Fractal Brownian Motion

This paragraph summarizes the mathematics behind fractal Black-Scholes option pricing, as presented in Li & Chen (2014). Difference between classical Brownian motion and fractal Brownian motion (fBM, denoted as \( B^H \)) comes from the covariance, where fBM may allows for serial correlation. \( B^H \) is defined as

\[
B^H(0, \omega) = 0 \\
B^H(t, \omega) = \frac{1}{\Gamma(H + \frac{1}{2})} \left[ \int_{-\infty}^{0} \left( (t-s)^{H-\frac{1}{2}} - (s-)^{H-\frac{1}{2}} \right) dB(s, \omega) + \int_{0}^{t} (t-s)^{H-\frac{1}{2}} dB(s, \omega) \right],
\]

with \( B(.) \) as classical Brownian motion, \( \Gamma(.) \) the gamma function and the Hurst exponent \( H \). \( H \) ranges from 0 to 1, \( H < 0.5 \) implies negative serial correlation, \( H > 0.5 \) positive one and \( H = 0.5 \) no autocorrelation, hence under \( H = 0.5 \) fBM equals classical Brownian motion.

Following, expectations of fBM are given by

\[
E[B^{H}_t] = 0 \quad \forall t \in R \\
E[B^{H}_t B^{H}_s] = \frac{1}{2} |t|^{2H} + |s|^{2H} - |t-s|^{2H}. \quad \forall t \in R \\
E[B^{H}_t]^2 = t^{2H} \quad \forall t \in R^+
\]

The price process under fractional geometric Brownian motion is then

\[
dS_t = \mu S_t dt + \sigma S_t dB^H_t,
\]

using \( S, \mu, \sigma \) as the stock price, fractal drift and fractal volatility. Li & Chen (2014) then show how applying (fractal) Itô Lemma yields

\[
d \ln(S_t) = \mu dt + \sigma dB^H_t - \frac{1}{2} \sigma^2 dt^{2H} \\
\ln \left( \frac{S_T}{S_0} \right) = \mu T - \frac{1}{2} \sigma^2 T^{2H} + \sigma B^H_T,
\]
so that the variance can be expressed through

\[
Var \left( \ln \left( \frac{S_T}{S_0} \right) \right) = E \left[ \ln \left( \frac{S_T}{S_0} \right)^2 \right] - E \left[ \ln \left( \frac{S_T}{S_0} \right) \right]^2 \\
= \left( \mu T - \frac{1}{2} \sigma^2 T^{2H} \right)^2 + 2\sigma \left( \mu T - \frac{1}{2} \sigma^2 T^{2H} \right) E \left[ B^H_T \right] + \sigma^2 E \left[ B^H_T \right]^2 \\
- \left[ \mu T - \frac{1}{2} \sigma^2 T^{2H} + \sigma E \left[ B^H_T \right] \right]^2
\]

which gives us Equation 4

\[
Var \left( \ln \left( \frac{S_T}{S_0} \right) \right) = \sigma^2 T^{2H},
\]

from which the option implied volatility surface can be decomposed.

**B Distributions**

![Hurst: implied vs. realized](image)

![Volatility: implied fractal vs. realized](image)

**Figure 9**

SP500: decomposition of ATM option’s implied volatility surface over time to maturity into its fractal components. Distribution of risk-neutral expectations vs. rolling 30-days realized measure. For both measures, distribution of investor expectations mainly overlays and comes close to the realized pattern.
Figure 10
SP500: decomposition of OTM option’s implied volatility surface over moneyness into its skewness and kurtosis components. Distribution of risk-neutral expectations vs. rolling 30-days realized measure. For both measures, distribution of investor expectations mainly overlays and comes close to the realized pattern.

C Volatility Prediction

Figure 11
Out-of-sample SP500 volatility prediction from machine learning algorithm building on option implied measures of fractal volatility, hurst exponent and skewness. The two shown cases demonstrate that the information content implied by options allow volatility prediction that come close to future realized ones. Problematic is that the longer the prediction horizon, the greater the lag of the model.