Risk Arbitrage Opportunities for Stock Index Options

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Abstract

To analyze the economic significance of pricing errors of stock index options, a system of linear inequalities is developed which completely characterizes all risk arbitrage opportunities which arise if a well-behaved pricing kernel does not exist. The Stochastic Arbitrage system can account for market imperfections in the form of transactions costs and general portfolio restrictions. An empirical methodology is proposed based on Empirical Likelihood estimation and Linear Programming. An active trading strategy based on the Stochastic Arbitrage system for front-month S&P500 stock index options yields significant abnormal returns out of sample, for small-scale portfolios. However, outperformance seems elusive if the strategy is scaled up and market depth is taken into account.

Key words: Options Pricing, Risk Arbitrage; Options Trading; Stochastic Dominance; Empirical Likelihood; Linear Programming

JEL codes: C61; C62; D53; D81; G11; G13

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1 Introduction

A large body of literature deals with the pricing and trading of financial options. Empirical problems encountered with standard models based on complete and perfect markets have motivated research which accounts for incompleteness and frictions.

These empirical problems include the option pricing kernel puzzle (Ait-Sahalia and Lo (2000) and Jackwerth (2000)). Stock index option prices imply a U-shaped pricing kernel, after the stock market crash of 1987, which challenges equilibrium asset pricing models based on a risk-averse representative investor.

Constantinides, Jackwerth and Perrakis (2009; CJP) forward an elegant and tractable system of linear inequalities for testing the existence of a well-behaved kernel function for pricing of a cross-section of concurrent stock index options in the presence of transactions costs.

The CJP system is motivated by the first-order optimality conditions for an investor who holds the underlying index. These optimality conditions ensure that the index is not dominated by any feasible alternative portfolio by the second-degree stochastic dominance (SSD) criterion (Hadar and Russell (1969); Hanoch and Levy (1969); Rothschild and Stiglitz (1970)).

The linear system is a natural generalization of SD pricing bounds for individual options in imperfect markets (Constantinides and Perrakis (2002, 2007)); the latter in turn generalize pricing bounds in frictionless markets (Perrakis and Ryan (1984); Levy (1985); Perrakis (1986); Ritchken (1985); Ritchken and Kuo (1988)).

CJP document widespread violations of the system for S&P500 stock index (SPX) options. These violations suggest that the option pricing kernel puzzle cannot be explained by accounting for incompleteness and frictions.

Motivated by these results, Constantinides, Czerwonko, Jackwerth and Perrakis (2011) and Constantinides, Czerwonko and Perrakis (2017) develop option trading strategies. The
former study considers strategies based on violations of pricing bounds for individual option contracts; the latter study builds portfolios which obey sufficient conditions for violation of the CJP system based on the ‘single-crossing rule’ for the cumulative distribution function (CDF) of the stock index.

The above studies form the quasi-totality of the empirical option studies that take into account the truly observed stock index option prices. All other empirical option market studies use models tailored to the frictionless market and modify the data to make it fit to their models.

The present study contributes to this literature by developing a complement to the CJP system which completely characterizes all portfolios which dominate the index by SSD. Compared with Constantinides, Czerwonko, Jackwerth and Perrakis (2011), the analysis is based on joint restrictions for a cross-section of option series instead of bounds for individual series; compared with Constantinides, Czerwonko and Perrakis (2017), the analysis uses necessary and sufficient conditions, which enlarges the set of risk arbitrage opportunities.

The relevant risk arbitrage opportunities are formally defined: a Stochastic Arbitrage (SA) opportunity arises when it is possible to engineer a self-financing overlay for the stock index which exhibits SSD, accounting for transactions costs and relevant portfolio constraints.

A linear system is developed to identify all SA opportunities if the option market is not in equilibrium. The SA system is a logical complement to the CJP system: an SA opportunity exists if and only if a well-behaved pricing kernel does not exist. This result extends the analysis by Dybvig (1988) from FSD and frictionless markets to SSD and imperfect markets.

The analysis is also reminiscent of the duality between the mean-standard deviation frontier for the pricing kernel and the mean-variance portfolio optimization problem (Hansen and Jagannathan (1991)). The use of SD instead of mean-variance dominance in the present study is motivated by the asymmetry of option returns.

The logical complementarity between pricing kernels and SA opportunities ties together the aforementioned studies of pricing and trading of stock index options using SSD. The
infeasibility of the CJP pricing kernel system lies at the root of the trading profits in Constantinides, Czerwonko, Jackwerth and Perrakis (2011) and Constantinides, Czerwonko and Perrakis (2017), something which these studies seem to have overlooked.

The SA system goes beyond the ‘simple’ application of linear duality theory to the CJP system. While the SA system describes all feasible and dominant solutions, the dual alternative to the CJP system gives only a necessary but not sufficient condition which even allows for unbounded positions. Furthermore, the optimal solution generally features multiple crossings of the CDF and, therefore, the SA system has more discriminatory power than the sufficient condition of Constantinides, Czerwonko and Perrakis (2017).

The SA system facilitates portfolio restrictions which are not binding for the index but which may become binding for active portfolios which include options, for instance, restrictions based on market depth in the limit order book. Such restrictions can enhance the realism of the analysis of the economic significance of pricing errors of stock index options.

The implementation of the SA system can draw on an existing literature on portfolio optimization with SSD constraints. The numerical optimization has been studied previously by Dentcheva and Ruszczyński (2003), Kuosmanen (2004), Roman, Darby-Dowman and Mitra (2006), Kopa and Post (2015) and Longarela (2016). Hodder, Jackwerth and Kolokolova (2015) previously applied these methods to an industry equity allocation problem.

For estimation of the return distribution of the underlying stock index, the present study relies upon the implied cumulative distribution function (ICDF) of Owen’s Empirical Likelihood (EL) method. The EL method combines the empirical cumulative distribution function (ECDF) with a set of moment conditions which capture side information while preserving the discrete nature of the distribution which facilitates numerical optimization.

In the asset pricing and portfolio optimization literatures, EL and related estimation methods have been employed previously by Almeida and Garcia (2012, 2016), Julliard and Ghosh (2012), Ghosh, Julliard and Taylor (2016), Post and Poti (2017) and Post, Karabati and Arvanitis (2017).
The emphasis here is on the physical distribution rather than the risk-neutral distribution, because the latter generally is not unique if the market is incomplete and does not exist if arbitrage opportunities exist. By using moment conditions which capture side information about market conditions, an estimate for the conditional return distribution is obtained.

In contrast to applications of SSD to a cross-section of primary assets or asset classes, the application of SSD to concurrent stock index options benefits from the natural single-factor payoff structure and the resulting ability to study a cross-section of individual securities without the curse of dimensionality.

In an empirical application to front-month SPX options, an active trading strategy based on the solutions to the SA system is evaluated. The investment performance depends critically on the options’ time to maturity and, furthermore, the assumed portfolio size. SA opportunities are larger for front-month options than for back-month options. Large-scale implementation does not outperform a passive position in the SPX, if market depth restrictions from the limit order book are taken into account. Consequently, the significant pricing errors found in previous research do not suffice to prove the existence of large-scale risk arbitrage opportunities.

2 Theory

2.1 Preliminaries

The investment universe includes $M$ securities: a Treasury bill, a stock index and $(M - 2)$ concurrent series of European options on the index. The focus is on self-financing solutions, so that new long positions are financed by selling existing positions in bills, stocks or options or, alternatively, by borrowing, short selling stocks or writing options.

The net purchase price (ask price plus trading fees) for a given security is denoted by $a_i \geq 0$; the net sales price (bid price minus trading fees) by $b_i$, $a_i \geq b_i \geq 0$; the value at
option expiration by \( x_i, i = 1, \cdots, M \).

The bill matures on the options expiration date and has principal \( x_1 \). The stock index pays cash dividends \( d \geq 0 \); the cum-dividend price is \( x_2 = y \) and the ex-dividend price is \((y - d)\). Options have strikes \( k_i \) and contingent payoffs \( x_i = \max(y - d - k_i, 0) \) for calls or \( x_i = \max(k_i - y + d, 0) \) for puts, \( i = 3, \cdots, M \).

No costs are involved for principal payments and cash settlements at the expiration date. Whether the index is traded at the expiration date depends on the assumed reinvestment and refinancing possibilities. For simplicity, it is assumed here that the unadjusted stocks position at expiration is carried to the next period, so that the end-of-period costs are zero. It is straightforward to adjust the cost structure in relevant cases, after deriving the SA system for the present specification.

A tractable discrete representation of the joint distribution of \( x = (x_1 \cdots x_M)' \) is used, to allow for the use of discrete mathematics and numerical optimization. The univariate payoff distribution of the index is represented by a multinomial, \( \mathcal{F} \), with atoms at \( y_j, y_1 < \cdots < y_N \), and physical probabilities \( p_j > 0, j = 1, \ldots, N \).

### 2.2 Pricing kernel system

The CJP system requires the existence of a non-negative and anti-monotonic pricing kernel, \( m(y): m(y_1) \geq \cdots \geq m(y_N) \geq 0 \). To facilitate compact presentation, the pricing kernel system in this study is presented both in summation notation and in matrix notation.

Using \( m_j, j = 1, \cdots, N \), for model variables which represent the values of the kernel, the formulation in summation notation (first introduced in Ritchken (1985)) follows:
\[
\sum_{j=1}^{N} p_j x_{ij} m_j \leq a_i, \ i = 1, \cdots, M; \\
- \sum_{j=1}^{N} p_j x_{ij} m_j \leq -b_i, \ i = 1, \cdots, M; \\
m_j - m_{j+1} \geq 0, \ j = 1, \cdots, N-1; \\
m_j \geq 0, \ j = 1, \cdots, N.
\]

The equivalent formulation in matrix notation uses \( P := (1_M \mathbf{p}), \ p := (p_1 \cdots p_N), \ X := (x_1 \cdots x_N), \ x_j := (x_{1j} \cdots x_{Mj}), \ \mu := (m_1 \cdots m_N)', \ a := (a_1 \cdots a_M)', \ b := (b_1 \cdots b_M)', \ D \) for an upper triangular difference matrix and the entrywise product ‘\( \circ \)':

\[
(P \circ X) \mu \leq a; \\
-(P \circ X) \mu \leq -b; \\
D \mu \geq 0_N; \\
\mu \geq 0_N.
\]

The system is equivalent to Constantinides, Jackwerth and Perrakis (2009, Eq. (1)-(9), p. 1253-1255), modulo the omission of end-of-period costs for trading stocks. It is straightforward to adjust the assumed investment problem after the corresponding SA system is derived.

The system can be interpreted as a set of first-order optimality conditions for a myopic and risk averse investor who is fully invested in the index or a position which is co-monotone to the index. The kernel represents the Intertemporal Marginal Rate of Substitution for this
The system is relatively small, because the kernel is evaluated only at the atoms of the distribution of the index, \( y_j, j = 1, \ldots, N \). These points are known in advance and are ranked \( (y_1 < \cdots < y_N) \), which allows for fixing the ranking of the kernel values \( (m(y_1) \geq \cdots \geq m(y_N)) \) and avoids the need to endogenize the ranking.

In practice, violations of the pricing conditions are pervasive for reasonable assumptions about transactions costs. A relevant question seems whether the violations are economically significant. The next section provides a complementary system which can be used to answer this question.

The analysis differs from Constantinides, Czerwonko, Jackwerth and Perrakis (2011) by building on the CJP system rather than the pricing bounds for individual options by Constantinides and Perrakis (2007); it extends Constantinides, Czerwonko and Perrakis (2017) by using necessary and sufficient conditions, rather than sufficient conditions, for SSD. The analysis therefore identifies a larger set of risk arbitrage opportunities than the previous studies. Notably, the optimal risk arbitrage portfolio violates the single-crossing rule for every cross-section of options in the empirical application.

### 2.3 Stochastic Arbitrage system

The investor benchmarks against a long position in the stock index and contemplates deviating from the index by constructing a layover portfolio. The decision variables are the numbers of securities bought at the ask price, \( \alpha \geq 0_M \), and the numbers of securities sold or written at the bid price, \( \beta \geq 0_M \); these numbers need not be natural numbers. The payoff to the layover is given by \( x = x'\alpha - x'\beta \); adding the layover to the index yields the enhanced portfolio payoff \( z = x + y \).

Examples of basic options strategies include the writing of covered calls and the buying of protective puts; more complex strategies combine multiple option types and strikes to build, for example, butterfly spreads and condors.
The feasible combinations for the layover portfolio are defined by a polytope of general form, \( \mathcal{P} := \{ (\alpha, \beta) \in \mathbb{R}^M \times \mathbb{R}^M : A\alpha + B\beta \leq c \} \), where \( A \) and \( B \) are the LHS and \( c \) are the RHS coefficients of the portfolio constraints. The solution \( (\alpha, \beta) = (0_M, 0_M) \) is assumed to be feasible and to have no binding constraints, so that \( (\epsilon 1_M, \epsilon 1_M) \in \mathcal{P} \) for sufficiently small \( \epsilon > 0 \). The portfolio set in the empirical application accounts for the market depth at the quoted best bid and ask prices.

SSD is used to provide a partial ordering of the feasible portfolios. SSD is formulated here in terms of the lower partial moment function \( L(x|t) := \mathbb{E}_F [(t - x) I(x \leq t)] \). Due to the use of a discrete distribution, this function is piecewise-linear: \( L(x|t) = \sum_{j=1}^{N} p_j (t - x_j) I(x_j \leq t) \).

**Definition 2.3.1 (SSD):** Payoffs \( z \) dominate payoffs \( y \) by SSD, or, \( z \succeq_{SSD} y \), if \( L(z|y_j) \leq L(y|y_j) \) for all \( y_j, j = 1, \cdots, N \).

Due to its functional properties, the lower partial moment function function needs to be evaluated only at the atoms \( y_j, j = 1, \cdots, N \).

A well-known equivalent formulation is that SSD occurs if the first alternative achieves a higher expected utility than the second alternative for all nonsatiable and risk averse investors.

Examples of options strategies aimed to create SSD of the index include the covered writing of call options and bull spreads and investing the option premium income in bills or stocks. These strategies sacrifice upside potential to reduce downside risk. Whether the premium income suffices to establish SSD of course depends on the prevailing bid and ask prices. The covered writing of butterfly or condor spreads also reduces downside risk but in exchange for lower returns during sideways markets instead of market upswings.

**Definition 2.3.2 (Stochastic Arbitrage):** A Stochastic Arbitrage opportunity arises if a self-financing layover \( x = x'\alpha - x'\beta \) can be formed such that net costs are negative \( (a'\alpha - b'\beta < 0) \) and the payoffs of the enhanced portfolio \( z = x + y \) dominate the payoffs of
the underlying index y by SSD:

\[ x'\alpha - x'\beta + y \geq_{SSD} y; \]

\[ a'\alpha - b'\beta < 0; \]

\[ (\alpha, \beta) \in P. \]

In contrast to the CJP system (2), the SA system (3) is not linear due the binary variables \( I(x_j \leq t), j = 1, \cdots, M, \) in the lower partial moment function. However, Section 3.2 provides an equivalent linear system using a known linearization of the lower partial moment. The linearized system is larger and computationally more demanding than the CJP system, but it also generates additional information.

The two systems can be shown to be logical complements:

**Theorem 2.3.3:** A Stochastic Arbitrage opportunity (or solution to (3)) exist if and only if no well-behaved pricing kernel (or solution to (2)) exists.

The two systems thus lead to identical conclusions regarding option market efficiency. However, in contrast to the CJP system (2), the SA system (3) defines all feasible SA opportunities, which is useful to analyze the economic significance of pricing errors and engineer option positions in practice.

The logical complementarity between pricing kernels and SA opportunities seems a missing link in the literature about pricing and trading stock index options using SSD. The empirical violations of the pricing kernel system documented by CJP lies at the root of the trading profits documented by Constantinides, Czerwonko, Jackwerth and Perrakis (2011) and Constantinides, Czerwonko and Perrakis (2017).

The SA system (3) is not simply the dual alternative to the CJP system (2). The proof to Theorem 2.3.3 in the appendix derives the dual alternative using Farkas’ Lemma; see Eq. (10). The dual alternative gives a necessary but not sufficient condition for dominance by
the solution portfolio, similar to the dual formulation of nonparametric portfolio efficiency
tests in Post (2003, Thm 2). Among other problems, the dual alternative does not bound
the option positions, allowing for unbounded short positions which generally introduce SSD
violations and insolvency risk.

Essentially, the CJP system does not account for dominance and feasibility restrictions
which are not binding for the index. Since these restrictions have a zero shadow price,
they do not affect the optimality classification of the index and they can be ignored for
testing the system of pricing restrictions. However, these restrictions may become binding
for enhanced portfolios and should therefore be taken into consideration when analyzing
economic significance of pricing errors if a well-behaved kernel does not exist.

The SA system (3) builds on the exact definition of SSD. By contrast, Constantinides,
Czerwonko and Perrakis (2017) use a single-crossing rule for the CDF \( F_y(t) := \mathbb{E}_X [I(y \leq t)] \),
which gives a sufficient but not necessary condition for SSD. The following corollary to
Definition 2.3.1 characterizes the tightness of the sufficient condition in the present study:

**Corollary 2.3.4 (Multiple-Crossing Rule):** If \( z \succeq_{SSD} y \), and, furthermore, \( L(z|y_2) <
L(y|y_2) \), \( L(z|y_j) = L(y|y_j) \), for some \( j = 3, \ldots, N - 1 \), and \( L(z|y_N) < L(y|y_N) \), then \( F_z(t) \)
and \( F_y(t) \) cross at least twice.

The optimal portfolio and market index in the empirical application invariably obey the
stated premises and, consequently, the two CDFs cross at least twice in every sample; see
Section 3.2.

Further research could extended the SA system to multi-period investment problems, to
complement the system of multi-period pricing restrictions in Appendix A of Constantinides,
Jackwerth and Perrakis (2009, p. 1272-1273). Naturally, there is no need to consider more
than one period if SA exist under the current definition for buy-and-hold strategies.
3 Methodology

3.1 Probability estimation

A variety of estimation methods could be used to obtain the physical probability vector $p$, depending on the nature of the available data and side information. The estimation of the univariate distribution of low-frequency returns of an aggregated index generally allows for non-parametric approaches. Non-parametric analysis notably avoids the traditional assumption of a lognormal distribution of the Black and Scholes model, which has limited descriptive validity for both the physical distribution of returns to typical stock indices and the risk-neutral distribution implied by index options.

A natural starting point is the ECDF for a representative time series of index returns. The ECDF assigns equal probability to all historical realizations. It is a statistically consistent, nonparametric Maximum Likelihood estimator, under appropriate stationary and dependence assumptions. More efficient estimators can be obtained using non-parametric General Method of Moments (GMM) or General Empirical Likelihood (GEL) methods which can account for side information in the form of moment conditions and possible dynamic patterns in the data.

The present study focuses upon Owen’s Empirical Likelihood (EL) method. EL estimates the probabilities by minimizing the Kullback-Leibler divergence to the equal-probability vector $q := N^{-1}1_N$ subject to $K$ empirical moment conditions. Using $g(y) : \mathbb{R} \rightarrow \mathbb{R}^K$ for the relevant vector-valued moment function, the moment conditions amount to $\mathbb{E}_F [g(y)] = Gp' = 0_K$, where $G := (g(y_1) \cdots g(y_N))$. The moment function may feature latent model parameters, nuisance parameters and, to obtain moment inequality conditions, slack variables.

Possible moment conditions include restrictions on the expected value, standard deviation and skewness of the index returns. Pricing restrictions for selected option series or options
combinations may also be included. Section 4.3 further elaborates on the specification of the moment function which is used in this study.

The EL implied probabilities are the solution for the model variables $\pi$ in the following Convex Optimization problem:

$$\min -q (ln(\pi)' - ln(q)')$$

$$G\pi^' = 0_K;$$

$$\pi 1_N = 1;$$

$$\pi^' \geq 0_N.$$  

The ICDF can be seen as a constrained nonparametric Maximum Likelihood estimator for the physical distribution function. Qin and Lawless (1995) establish statistical consistency, asymptotic normality and semi-parametric efficiency of this estimator for serially Identical and Independently Distributed data.

The blockwise implementation by Kitamura (1997) applies for a class of stationary and weakly dependent processes. This approach is particularly useful for high-frequency returns, which involve long time series and strong dynamic effects. Due to the focus on low-frequency returns, the blockwise approach however has a limited advantage in the present application, notwithstanding the evidence for mean reversion in stock market volatility.

The ECDF is sparse in the tails of the distribution. To improve the estimation of the tails, smoothed versions of the ECDF can be employed, for example, using kernel density estimation. The above methodology can then be applied to a discretized version of the estimated distribution.

This improvement is particularly relevant for analyzing deep-out-of-the-money options. However, the importance of these options in the present study is limited due to their high

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bid-ask spreads and limited market depth. The use of kernel density estimation therefore has a limited effect in the present study. Caution is also advised to avoid look-ahead bias in the selection of the optimal bandwidth and kernel function.

The benefits of refinements of the probability estimation depend on the relevant set of portfolio constraints. It is well known that investment restrictions tend to mitigate the effects of estimations error and yield more robust investment performance; see, for instance, Jagannathan and Ma (2003) and DeMiguel, Garlappi, Nogales and Uppal (2009).

3.2 Numerical optimization

The existence of a feasible solution can be determined using numerical optimization. A literature in OR/MS has developed various linear formulations of the SSD relation for discrete distributions; see Kopa and Post (2015) for a survey and generalization.

Using \( L := (L(y|y_1) \cdots L(y|y_N))' \) and \( y := (y_1 \cdots y_N) \), the following linearization is obtained:

**Proposition 3.2.1:** Every SA is a solution to the following system of linear inequalities and every solution is an SA opportunity:

\[
\begin{align*}
    p\Theta & \leq L; \\
    -\Theta - X'\alpha 1_N + X'\beta 1_N' & \leq y'1_N - 1_N y; \\
    a'\alpha - b'\beta & < 0; \\
    A\alpha + B\beta & \leq c; \\
    \alpha, \beta & \geq 0_N; \\
    \Theta & \geq 0_{N \times N}.
\end{align*}
\]
The linear system can be tested using Linear Programming (LP) techniques. Notably, an LP problem can be designed for optimizing an objective function subject to the dominance and feasibility constraints (5). The relevant specification of the objective function depends on the application at hand.

An appealing problem specification is to maximize current income \((b'\beta - a'\alpha)\) subject to the linear system:

\[
\begin{align*}
\max & \quad b'\beta - a'\alpha \\
\text{s.t.} & \quad p\Theta \leq L; \\
& \quad -\Theta - X'\alpha 1'_N + X'\beta 1'_N \leq y'1'_N - 1_N y; \\
& \quad A\alpha + B\beta \leq c; \\
& \quad \alpha, \beta \geq 0_N; \\
& \quad \Theta \geq 0_{N \times N}.
\end{align*}
\]

If an SA opportunity exists, then the optimal overlay generates current income of \(b''\beta - a''\alpha > 0\) and the enhanced portfolio payoff at option expiration dominates the index payoff by SSD: \(x'\alpha^* - x'\beta^* + y \succeq_{SSD} y\).

The above problem specification tends to improve the robustness of total holding period return, because current income is not conditional on the index and not affected by sampling error. Current income in effect shifts up the entire return distribution by a constant and thus operates as a slack variable for the dominance constraints.

Robustness may be further enhanced by appropriate relaxations of the SA system. Due to the complexity of the SD relation, sampling error has an asymmetric effect: false dominance classifications are less likely to occur than false non-dominance classifications; see
Kroll and Levy (1980) and Linton, Maasoumi and Whang (2005). This pattern suggests that relaxing instead of tightening the inequalities can enhance out-of-sample performance; various alternative relaxations are discussed in the Conclusions.

The single-crossing rule used in Constantinides, Czerwonko and Perrakis (2017) tightens the optimization problem instead of relaxing it. In our empirical application, the premises of Corollary 2.3.4 are satisfied, in every sample. Notably, the optimization ensures that some of the risk constraints are binding in the interior of the support: \( L(x'\alpha^* - x'\beta^* + y|y_j) = L(y|y_j) \), for some \( j = 3, \ldots, N - 1 \). Consequently, the CDFs of the optimal portfolio and the market index cross at least twice. This finding implies that the single-crossing rule invariantly leads to suboptimal solutions.

Since \( N \gg M \) in typical applications, the number of variables and constraints is \( O(N^2) \) and increases at a quadratic rate with the number of grid point for partitioning the support of the underlying distribution. However, the problem is perfectly tractable for hundreds of partition points with standard computer hardware and software. For an ultra-fine partition, high-performance platforms and specialized solver software are recommended.

4 Analysis of SPX Options

4.1 Data

A panel data set is used from CBOE’s DataShop containing all available best bid and ask prices and sizes for European-style call and put options on the S&P 500 stock index (SPX) recorded at 14:45 CT (30 minutes before the close at 15:15 CT) from April 1, 2004, through April 1, 2018. In addition, the analysis uses daily SPX values (source: Yahoo Finance), total monthly dividends on the index (source: Robert Shiller’s homepage) and T-bill rates and dollar-denominated LIBOR rates (source: Federal Reserve Bank of St. Louis).

Finally, the analysis uses daily values to ten CBOE Strategy Benchmark Indices for known trading strategies using SPX options: (1) BuyWrite Index (BXM), (2) PutWrite
Index (PUT), (3) 95-110 Collar Index (CLL), (4) 2% OTM BuyWrite Index (BXY), (5) 30-Delta BuyWrite Index (BXMD), (6) Iron Butterfly Index (BFLY), (7) Zero-Cost Put Spread Collar (CLLZ), (8) Covered Combo Index (CMBO), (9) Iron Condor Index (CNDR), (10) 5% Put Protection Index (PPUT).\(^1\)

Except from using a more recent sample period, the empirical design resembles that of CJP; 28 days before each expiration date, the SA problem (6) is solved for the cross-section of front-month options. The sample period includes 168 formation and expiration dates.

A maturity of 28 days seems appropriate here, for a number of reasons. Longer maturities seem less relevant, because back-month options are known to have relatively smaller pricing errors. Furthermore, shortening the maturity below 28 days implies that the enhanced portfolio is passive (that is, it does not have an open position in options) for a larger part of the month. Nevertheless, the analysis is extended to consider also active strategies based on 91-day options and 7-day options.

Figure 1 show that hundreds of different option series are available on the typical formation date. The number of available series \((M - 2)\) determines the number of decision variables in the SA problem and the scope for financial engineering of option combinations.

[Insert Figure 1 about here.]

Data filters based on liquidity and moneyness have a minimal effect in this study, because bid-ask spreads and market depth are explicitly taken into account in the portfolio constraints; see Section 4.2.

Figure 2 illustrates the evolution of the quoted volumes during the sample period. Every panel shows the quartile break points (across the available put and call option series) of the combined bid and ask quote size, for a given maturity (7 days, 28 days or 91 days). The typical quote size ranges from many tens to some hundreds of contracts, with higher levels

and larger fluctuations in the period 2009-2015. Exceptionally large cross-sectional variation in quote size is seen in 2016-2017. Not shown in the figure is that average ask quote size significantly exceeds average bid ask size, especially for 7-day options, and that average put quote size significantly exceeds average call quote size.

[Insert Figure 2 about here.]

At the expiration day, the payoff to the enhanced portfolio is computed as the dividend-adjusted value of the index plus the net payoff of the overlay portfolio. The total holding period return is computed as this payoff plus the initial income of the overlay with accrued interest.

In addition to excess returns, the analysis also considers abnormal returns relative to the index and abnormal returns computed using Returns Based Style Analysis (RBSA; Sharpe (1988)) based on the aforementioned options strategy indices. The RBSA helps to diagnose the implied investment strategy and determine which part of portfolio returns may be attributable to exposure to option-specific risk factors.

It should be stressed that the option strategy indices do not account for transactions costs and investment restrictions and hence may not represent feasible investment returns. Furthermore, the option strategy indices underperform the passive SPX index in the evaluated sample period, even without accounting for transactions costs, which limits the effect of the RBSA on the estimated outperformance.

The latent CDF is estimated using the ECDF as well as using the EL ICDF with moment conditions based on the market prices of at-the-money puts and calls; see Section 4.3 below. Since the ECDF does not use conditioning information, a separate analysis is performed without the 2007-2009 period, which features a bear stock market and large swings in general market volatility during the Global Financial Crisis (GFC).
4.2 Portfolio constraints

The multiplier of the SPX options is $100 per index point and the average value of the index over the sample period is equal to 1,546. Hence, a unit position in the index represents an average investment of $154,600. This scale may represent a retail portfolio. Since overhead costs are not taken into account, the analysis will overstate the net profitability to investors who face costs for management, facilities and Information and Communications Technology infrastructure.

To capture portfolios of institutional investors, initial positions of 10, 100 and 1,000 times the index are also considered, which corresponds to portfolios of $1,55m, $15.5m and $155m, respectively.

A larger scale naturally requires a larger transaction volume and thus involves stronger trade impact and higher slippage risk. To capture this effect, position limits are imposed based on the depth of the price quotes: \( \alpha \leq s(a) \), \( \beta \leq s(b) \), where \( s(a) \) and \( s(b) \) are the quoted best ask size and best bid size, respectively.

These constraints are permissive in the sense that the quoted volume may no longer be available when the buy or sell orders are placed (slippage risk). On the other hand, the constraints are restrictive in the sense that the volumes offered at less favorable prices are not considered, because the data set includes only the best bid and ask prices and Level-II data about the order book were not available for this study.

The market depth constraints differ in a subtle way from the position limits in Constantinides, Czerwonko and Perrakis (2017) who restrict the total size of the option component to one option per one unit of index, regardless of quoted volumes. The market depth restrictions are not scalable: they have no material effect for the unscaled portfolio but become highly relevant for the scaled portfolios (10, 100 or 1,000 times the index).

Trading fees in addition to the bid-ask spread are not explicitly included in the optimization process. Instead, break-even fees are computed after the optimization by computing
the ratio of the net initial income of the layover to the total value of the traded options. These break-even fees can be interpreted as the level of proportional trading fees that would eliminate the SA opportunities.

Solvency constraints and margin constraints have a minimal effect on the analysis because the SSD constraints force the solution portfolio to have less downside risk than the index. Among other things, SSD requires that the solution portfolio has a higher minimum return than the index, which implies that the enhanced portfolio is solvent in every scenario: $X'\alpha^* - X'\beta^* \geq -y$.

To the extent that margin requirements are designed to ensure solvency of the writer, the SSD restrictions thus reduce the need for explicit margin requirements. By contrast, the dual alternative to the CJP system (which does not give a sufficient condition for SSD by the solution portfolio) does not ensure solvency and would benefit from adding explicit solvency constraints or margin requirements.

4.3 Probability estimation

The physical probabilities are estimated using the ECDF based on an expanding window consisting of maximally overlapping 28-day excess returns over the period from January 1954 through the relevant formation date. Kernel density estimation is used to smooth the ECDF, using a normal kernel function and Silverman’s rule for bandwidth selection.

To allow for numerical optimization, the smoothed ECDF is discretized by partitioning the empirical return range of the SPX using a finite number of bins of equal width and integrating the probability mass in every bin.

The ECDF can be seen as an estimate for the unconditional return distribution. To obtain conditional estimates, EL implied probabilities are computed based on side information at every formation date.
The side information consists of the market prices of the put and call options \((i = 3, \cdots, M)\) which are closest to being at-the-money. These options have relatively high liquidity and can be assumed to be fairly priced, an assumption which is supported by their low average abnormal returns out of sample. Their prices can therefore be used to estimate the conditional distribution.

To avoid the specification of a pricing kernel and allow margin for pricing errors, general model-free pricing bounds are used. Regretfully, the pricing bounds by Constantinides and Perrakis (2002, 2007) are non-convex functions of the probabilities, which complicates their inclusion in moment conditions. For this reason, alternative, linear pricing bounds are developed.

The following pricing bound for call options is used as a first moment condition:

\[
\mathbb{E}_F \left[ x_i \right] \frac{a_1}{x_1} \leq b_i.
\] (7)

This bound is motivated by the inequality \(\mathbb{E}_F \left[ x_i \right] (a_1/x_1) \geq \sum_{j=1}^{N} p_j x_{ij} m_j\), which, in turn, is based on (i) antimonotonicity of call payoff \(x_i = \max(y-d-k_i,0)\) and the kernel, and (ii) pricing restriction \(\sum_{j=1}^{N} p_j m_j \leq (a_1/x_1)\) for the riskless asset.

Similarly, monotonicity of put payoff \(x = \max(k_i - y+d,0)\) and the kernel motivates the following bound for put options is used as a second moment condition:

\[
\mathbb{E}_F \left[ x_i \right] \frac{b_1}{x_1} \leq a_i.
\] (8)

The relevant moment function for imposing the above moment conditions is \(g_i(y) = (a_1/x_1) x_i - b_i - s_i\) for calls and \(g_i(y) = (b_1/x_1) x_i - a_i + s_1\) for puts, where \(s_i \geq 0\) is a slack variable which transforms the general moment equality condition \(\mathbb{E}_F [g_i(y)] = 0\) to an inequality condition.
Roughly speaking, the moment conditions increase the probability assigned to large gains (losses) above their historical relative frequency, which increases the calculated value of the call (put) options, when the market prices of the call (put) options are elevated due to, for instance, a higher expected upside potential (downside risk) for the market index.

The implied probabilities are estimated using only historical returns and current market prices which are available on the formation date; the out-of-sample investment performance is therefore free of forward-looking bias.

4.4 Statistical testing

Statistical tests can be used to determine whether the out-of-sample dominance relations between the enhanced portfolio and the index are statistically significant.

Regretfully, small-sample tests for SSD are difficult to obtain due to the model-free assumption framework and the complexity of the hypothesis structure. However, several large-sample tests are available.

Most statistical tests use a null hypothesis of dominance and an alternative of non-dominance. This hypothesis structure seems well suited for the evaluation of out-of-sample returns of portfolios which are constructed to dominate the index.

The present study focuses on the Linton, Maasoumi and Whang (2005) test for SSD, which uses a bootstrap method to estimate critical values for a Kolmogorov-Smirnov type test statistic. In the (rare) case of a perfect out-of-sample dominance relation, the LMW05 test statistic takes a value of zero and the p-value is estimated to take the value of one.

If dominance cannot be rejected, it becomes relevant to also test for non-dominance, to avoid false dominance classifications due to a possible lack of statistical power of the SSD test. For this purpose, the bootstrap Empirical Likelihood Ratio test by Davidson (2009) is used, with cutoffs at the first and ninth decile break points of the distribution of out-of-
sample SPX returns. If empirical violations of SSD occur for return levels which lie between the two cutoffs, then the D09 test statistic takes a value of zero while the LMW05 test statistic takes a strictly positive value.

4.5 Results

Table I summarizes the out-of-sample investment performance of the passive strategy and the enhanced portfolio for various scales. The focus is on using 28-day options, the conditional CDF estimator (ELICDF) and the entire sample period. Results for alternative specifications will be presented and discussed later on.

For the unscaled strategy \((S = 1)\), SA opportunities which generate current income in excess of one index point ($100) are found for 163 out of 168 formation dates. A buy-and-hold investor in the index can substantially improve her performance by adding an active options overlay. The average excess return of the passive portfolio can be boosted by an impressive 36.34% per annum.

The annualized Information Ratio (IR) takes a value of 0.54. Although this value is impressive given the relatively low rebalancing frequency, it materially understates the economic appeal of the enhanced portfolio, as it does not account for the pronounced asymmetry of return distribution.

The return enhancement is achieved without significant additional downside risk and by significant improvements in upside potential, witness the high positive skewness of 9.88. An almost perfect out-of-sample dominance relationship is achieved, with only minimal empirical violations of SSD. The LMW05 test cannot reject dominance (p-value: 0.76). Since minor empirical violations of dominance occur, the D09 test takes a value of zero (p-value: 1.00).

The break-even fees take an average value of about 9%. It therefore seems unlikely that neglected fees in addition to the bid-ask spread can explain away the SA opportunities.
The results of the RBSA reveal that only a small part of the returns is explained by the exposures to known option strategies. The enhanced portfolio has a loading of 0.14 for the BXMD index (covered calls), 0.77 for CNDR (condors) and zero for the other eight indices; the R-squared is limited to about 2 percent. The conclusion is that there exist no indications that the high average return is compensation for exposure to option-specific risk factors.

The above results are consistent with those in the aforementioned earlier studies of active options trading strategies. Unfortunately, the results appear not robust to the assumed portfolio size. Many of the optimal option positions are equal to or approach the quoted bid or ask size for the relevant option series, which raises concerns about the scalability of the strategy. Indeed, the investment performance quickly deteriorates as the scale of the portfolio is increased. Scaling the portfolio reduces the economic and statistical significance of outperformance and eventually leads to underperformance.

Already for 100 times the index ($S=100$), or a $15,46$m portfolio, the enhanced portfolio underperforms the index out of sample. The LMW05 test rejects dominance at conventional significance levels; the D09 test of course confirms that there exists no empirical evidence non-dominance.

Since the analysis ignores trading fees and slippage risk, the profitability of the large-scale strategies is likely to be even lower in practice than is estimated here.

[Insert Table I about here.]

Figure 3 and Figure 4 illustrate the typical portfolio composition of the options layover portfolio. Every panel ($S=1, 10, 100, 1000$) shows the quartile break points (across the formation dates) for the number of options at a given level of moneyness ($S/X$). For many moneyness levels, the median position is close to zero and the distribution is highly skewed to negative values (short positions) or highly skewed to positive values (long positions).

For calls (Figure 3), the average position resembles a short call butterfly spread (write
one out-of-the-money call, buy two at-the-money calls, write one in-of-the-money call). Since the quoted volume is relatively high at and near the money, the call positions are relatively robust to scaling and the four panels show a similar pattern.

For puts (Figure 4), the average position resembles a long put bear spread (buy puts with a given strike write puts which a lower strike). For small-scale portfolios, the put spreads are (deep) out of money. Due to the relatively low quoted volume of the relevant puts, these positions are not scalable. Consequently, for large-scale portfolios, at-the-money put spreads are added.

Figure 5 further illustrates the results. Every panel demonstrates the typical conditional risk profile of the enhanced portfolio by plotting the quartile break points (across all expiration dates) of the investment return against the index return. The figure uses the feature that the returns to all option series are fully determined by the index return.

The most noticeable pattern is a strong outperformance when the SPX crashes by 20-30%. This pattern reflects the large number of bear spreads. As the portfolio scale increases and it becomes increasingly more difficult to find sufficiently many mispriced options, the risk profile more closely resembles that of the SPX index but the emphasis on systematic downside risk reduction remains.

Figure 6 illustrates the out-of-sample dominance relation by comparing the out-of-sample ECDF of the enhanced portfolio with that of the SPX. The large increase in mean return is
clearly visible and a near-perfect out-of-sample dominance relation emerges, for the unscaled strategy \((S = 1)\). As the portfolio scale increases, the mean quickly deteriorates and large violations of dominance occur.

[Insert Figure 6 about here.]

The results are surprisingly robust to the exclusion of the 2007-2009 period with extreme market volatility levels. The robustness is attributable to the use of market information by the conditional estimator. If the unconditional CDF estimators (ECDF and smoothed ECDF) are used, then results become very sensitive of the sample period.

During the GFC, market volatility and options prices climbed to unprecedented levels. Using an unconditional CDF estimator, option prices appear very expensive, providing an strong incentive for writing options. Since unconditional CDF estimators assign minimal probability to double-digit losses, the optimal enhanced portfolio writes deep-out-of-the-money put options on a large scale. This strategy is very profitable but also extremely risky. Despite the very high dispersion, the strategy is quite fortunate during the GFC: it narrowly escapes large losses during the largest market down swings and generates a bumper profit in the month of December 2008 when market volatility unexpectedly fell from historical highs.

The analysis is further extended by changing the time to expiration of the traded options. Consistent with Constantinides, Czerwonko and Perrakis (2017), the profitability of the strategy falls when back-month options are considered. Table II shows the results for three-month options. The investment performance deteriorates significantly compared with that based on the 28-day options in Table I by every performance measure. The SSD relation is now only marginally significant for the unscaled portfolio, using the LMW05 test.

[Insert Table II about here]
Maturities shorter than 28 days are more profitable over the life the options, but the overall monthly returns are still materially smaller than those of one-month options because the enhanced portfolio is not invested in options for a sizable part of the month. Table III shows the results for 7-day options, assuming that the portfolio is fully invested in SPX when 7-day options are not available. The RBSA reveals that the enhanced portfolio more closely resembles known option trading strategies of writing covered calls, butterfly spreads, strangles and condors, especially is the portfolio is scaled up \((S = 10, 100, 1000)\).

[Insert Table III about here]

5 Conclusions

The CJP system (2) amounts to a set of first-order optimality conditions for Convex Optimization. These conditions are computationally efficient for testing the optimality of the passive stock index but they do not identify the dominating alternatives if the index is suboptimal. By contrast, the larger SA system captures all feasible risk arbitrage opportunities and thus better captures the economic significance of pricing errors if the market is inefficient.

The empirical application shows that systematic SA opportunities exist for front-month SPX options, consistent with the conclusions of CJP, Constantinides, Czerwonko, Jackwerth and Perrakis (2011) and Constantinides, Czerwonko and Perrakis (2017). However, the economic significance seems to depend critically on the assumed portfolio size. Ex-ante outperformance seems elusive for large-scale implementation of the proposed trading strategy, if market depth is taken into account. In fact, the large-scale strategy underperforms a passive position in the SPX out of sample, due to the bid-ask spreads.

A number of extensions are left for further research. To further analyze the scalability of SA opportunities, Level II data about the order book could be used to assess the price impact.
of large transactions. This approach would account for the volumes offered at prices above the best ask prices and requested at the best bid prices. The large performance deterioration observed in this study for the relatively small scaling factor of \( S = 10 \) however mitigates the expectations for finding substantially more 'money left on the table'.

The definition of SA in this study is based on the SSD relation. Additional risk arbitrage opportunities may be uncovered using third-degree stochastic dominance (TSD) which assumes decreasing risk aversion in addition to global risk aversion. Restricting the permissible risk preferences amounts to relaxing the SA system, because it enlarges the set of dominating portfolios. Post and Kopa (2017) develop Quadratic Programming solutions for portfolio optimization with TSD constraints.

Other possible relaxations of the SA system include the use of Almost Stochastic Dominance (Leshno and Levy (2002)) and the weakening of dominance constraints in empirical ‘contact areas’ (Linton, Post and Whang (2014)). Relaxing instead of tightening the SA system seems in order because sampling variation tends to fuel false non-dominance classifications rather than false dominance classifications (Kroll and Levy (1980)).

References


Appendix

Proof to Theorem 2.3.3: The strategy is to first derive the dual alternative to system (2) and next to show that the alternative has a solution if and only if the system (3) is solvable.

To allow for compact notation, let $\mathbf{P} := (1_M \mathbf{p})$ and $\mathbf{C} := (\mathbf{P} \circ \mathbf{X}) \mathbf{U}$, where $\mathbf{U}$ is an upper triangular summation matrix. Let $\gamma = \mathbf{D}\mu$ be the decrements of the kernel, so that $\mathbf{U}\gamma = \mu$. System (2) can be reduced to

\[
\begin{align*}
\mathbf{C}\gamma & \leq \mathbf{a}; \\
-\mathbf{C}\gamma & \leq -\mathbf{b}; \\
\gamma & \geq 0_N.
\end{align*}
\] (9)

By Farkas’ Lemma (Gale, Kuhn and Tucker (1951, Lemma 1, p. 318)), a solution to the following alternative exists if and only if system (9) is not solvable:

\[
\begin{align*}
\mathbf{C}'\alpha - \mathbf{C}'\beta & \geq 0_N; \\
\mathbf{a}'\alpha - \mathbf{b}'\beta & < 0; \\
\alpha, \beta & \geq 0_M.
\end{align*}
\] (10)
This system requires the existence of an overlay portfolio which features negative net costs \((a'\alpha - b'\beta)\) and non-negative partial co-moments \(C_j' (\alpha - \beta) = C (x' (\alpha - \beta) | y_{j}),\)
\(j = 1, \cdots, N,\) where \(C (x|t) := \mathbb{E}_x [x \mathcal{I} (y \leq t)].\)

Every solution \((\alpha, \beta)\) to system (3) is also a solution to system (10), regardless of the specification of \(\mathcal{P}\). To see this, let \(x := x' \alpha - x' \beta\) and rewrite the lower partial moment inequality \(L (x + y|y_{j}) \leq L (y|y_{j}), \ j = 1, \cdots, N,\) as

\[
L(y|y_{j}) - C(x|y_{j}) + R (x, y|y_{j}) \leq L (y|y_{j}),
\]

where \(R (x, y|y_{j}) := \mathbb{E}_x [(y_{j} - x - y) (\mathcal{I} (x + y \leq y_{j}) - \mathcal{I} (y \leq y_{j}))].\) Due to co-monotonicity of \((y_{j} - x - y)\) and \((\mathcal{I} (x + y \leq y_{j}) - \mathcal{I} (y \leq y_{j}))\), \(R (x, y|y_{j}) \geq 0\), and (11) implies the co-moment inequality \(C (x|y_{j}) \geq 0\), which suffices to prove that that system (10) is solvable if system (3) is solvable.

Furthermore, every solution \((\alpha, \beta)\) to system (10) implies the existence of a solution \((\epsilon \alpha, \epsilon \beta), \ \epsilon > 0,\) to system (3). Let \(x_{j} := x' \alpha - x' \beta, j = 1, \cdots, N.\) Since \(y_{1} < \cdots < y_{N}\), there exists \(\epsilon > 0\) which is sufficiently small to ensure that \(\epsilon x_{j} + y_{j} < y_{k}\) and \(y_{j} < \epsilon x_{k} + y_{k}\) for all \(j = 1, \cdots, N - 1,\) and \(k > j.\) For this small value of \(\epsilon > 0,\) \(\mathcal{I} (\epsilon x_{k} + y_{k} \leq y_{j}) = \mathcal{I} (y_{k} \leq y_{j})\) for all \(k \neq j.\) Consequently, the following equality applies for every \(j = 1, \cdots, N:\)

\[
L(\epsilon x + y|y_{j}) = L(y|y_{j}) - S (\epsilon x|y_{j}),
\]

where \(S(x|y_{j}) := \min (0, C(x|y_{j}))\) and \(S(x|y_{j}) := \min (C(x|y_{j+1}), C(x|y_{j})), j = 2, \cdots, N.\)

The co-moment inequalities \(C (x|y_{j}) \geq 0, \ j = 1, \cdots, N,\) imply \(S(x|y_{j}) \geq 0 \Rightarrow S (\epsilon x|y_{j}) \geq 0, \ j = 1, \cdots, N,\) and therefore (via (12)) the lower partial moment inequalities \(L(\epsilon x + y|y_{j}) \leq L(y|y_{j}),\) for sufficiently small \(\epsilon > 0.\) In addition, feasibility, or \((\epsilon \alpha, \epsilon \beta) \in \mathcal{P},\) can be established for sufficiently small \(\epsilon > 0,\) because the portfolio constraints are not binding at \((\alpha, \beta) = (0_{M}, 0_{M}).\) \(\blacksquare\)

**Proof of Corollary 2.3.4:** Let \(D_{1}(z, y|t) := F_{z}(t) - F_{y}(t)\) and \(D_{2}(z, y|t) := L(z|t) - L(y|t).\) Since \(F_{z}(t) = \partial L(x|t)/\partial t,\) it follows that \(D_{1}(z, y|t) = \partial D_{2}(z, y|t)/\partial t.\) The stated conditions imply that \(D_{2}(z, y|y_{1}) = 0\) (by \(z \geq_{\text{SSD}} y\) and \(L(y|y_{1}) = 0),\) \(D_{2}(z, y|y_{2}) < 0\) and \(D_{2}(z, y|y_{N}) < 0.\) Therefore, if \(D_{2}(z, y|y_{j}) = 0,\) for some \(j = 3, \cdots, N - 1,\) then it follows that \(D_{2}(z, y|t)\) has at least one local maximum and at least one local minimum, and, furthermore, the derivative \(D_{1}(z, y|t)\) changes sign at least twice. Since \(D_{1}(z, y|t) = F_{z}(t) - F_{y}(t),\) this situation requires at least two CDF crossings. The condition \(L(z|y_{N}) < L(y|y_{N}),\) or \(D_{2}(z, y|y_{N}) < 0,\) avoids the situation in which \(D_{2}(z, y|t) = 0\) for all thresholds after the first local minimum, which would amount to a contact point instead of a crossing point. \(\blacksquare\)

**Proof of Proposition 3.2.1:** The SSD restriction \(x' \alpha - x' \beta + y \geq_{\text{SSD}} y\) amounts to \(L (x' \alpha - x' \beta + y|y_{j}) \leq L (y|y_{j})\) for all \(y_{j}, k = 1, \cdots, N.\) Using \(L (x + y|y_{j}) = \mathbb{E}_x (y - x - y)_{+},\)
where \((\cdot)_{+}\) is the element-wise positive part operator, the \(N\) lower partial moment constraints can be stacked as follows:

\[
p (1_{T} y - X' \alpha l_{1_{N}} + X' \beta l_{1_{N}} - y' l_{1_{N}})_{+} \leq L.
\]
Using a linearization based on Rockafellar and Uryasev (2000) and Dentcheva and Ruszczynski (2003), this system can be rewritten as

\[
\begin{align*}
 p\Theta & \leq L; \\
 -\Theta - X'\alpha'1_N + X'\beta'1_N & \leq y'1_N - 1_Ny; \\
 \Theta & \geq 0_{N\times N}.
\end{align*}
\] (14)

The additional model variables \( \Theta \) capture the element-wise positive parts; the solution \( \Theta = (1_Ty - X'\alpha'1_N + X'\beta'1_N - y'1_N)_+ \) is feasible if (13) is satisfied. Replacing (13) with (14) in (3) gives (5).
Table I: Out-of-sample performance of active strategies using one-month options

This table summarizes the results for 28-day options. The passive strategy is fully invested in the SPX. Active strategies are based on the ICDF estimator and portfolio-weight constraints for scale $S = 1, 10, 100, 1000$. The first panel examines percentage returns in excess of the LIBOR; the second panel evaluates returns in excess of the SPX; the third panel presents results of a Return Based Style Analysis based on 10 CBOE option strategy indexes. Finally, p-values of two statistical tests are displayed in the bottom panel: Linton, Maasoumi and Whang (2005) and Davidson (2009). Means, standard deviations, Sortino ratios and alphas are annualized.

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<td></td>
<td>$S = 1$</td>
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<tr>
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<td></td>
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<td></td>
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Table II: Out-of-sample performance of active strategies using three-month options

This table summarizes the results for 91-day options. The passive strategy is fully invested in the SPX. Active strategies are based on the ICDF estimator and portfolio-weight constraints for scale $S = 1, 10, 100, 1000$. The first panel examines percentage returns in excess of the LIBOR; the second panel evaluates returns in excess of the SPX; the third panel presents results of a Return Based Style Analysis based on 10 CBOE option strategy indexes. Finally, p-values of two statistical tests are displayed in the bottom panel: Linton, Maasoumi and Whang (2005) and Davidson (2009). Means, standard deviations, Sortino ratios and alphas are annualized.

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Table III: Out-of-sample performance of active strategies using one-week options.

This table summarizes the results for 7-day options. The passive strategy is fully invested in the SPX. Active strategies hold the SPX for 28 days and enhance this position with the ICDF-based optimal portfolio of options during the last week of this period. Portfolio scales $S = 1, 10, 100, 1000$ are used. The first panel examines percentage returns in excess of the LIBOR; the second panel evaluates returns in excess of the SPX; the third panel presents results of a Return Based Style Analysis based on 10 CBOE option strategy indexes. Finally, $p$-values of two statistical tests are displayed in the bottom panel: Linton, Maasoumi and Whang (2005) and Davidson (2009). Means, standard deviations, Sortino ratios and alphas are annualized.

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Figure 1: Cross-sectional dimension. The figure shows the total number of SPX option series available with a given maturity for every formation date during the sample period. The solid line represents the number of different strikes for calls options; the dashed line represents puts. The three different panels refer to three different maturities: 91 days, 28 days and 7 days.
Figure 2: Distribution of quote size. Shown are the quartile break points (across the available put and call option series) of the combined bid and ask quote size at every formation date in the sample period, for a given maturity. The three different panels refer to three different maturities: 91 days, 28 days and 7 days. The solid line represents the median; the dashed lines are the 25th and 75th percentiles.
Figure 3: Optimal number of calls. Shown are the quartile break points (across all formation dates) for the number of call options in the optimal overlay portfolio at a given level of moneyness \((S/X)\), for the analysis of 28-day SPX options. Each panel presents one of the four scaling factors: \(S = 1, 10, 100, 1000\). The solid line represents the median; the dashed lines are the 25th and 75th percentiles.
Figure 4: Optimal number of puts. Shown are the quartile break points (across all formation dates) for the number of put options in the optimal overlay portfolio at a given level of moneyness ($S/X$), for the analysis of 28-day SPX options. Each panel presents one of the four scaling factors: $S = 1, 10, 100, 1000$. The solid line represents the median; the dashed lines are the 25th and 75th percentiles.
Figure 5: Conditional risk profile of the enhanced portfolio. This figure displays three quartile break points for ex-ante returns of the enhanced portfolios with 28-day options. Each graph presents one of the four scaling factors: $S = 1, 10, 100, 1000$. The solid line represents the median (across all expiration dates) of the active return conditional on the SPX; the dashed lines correspond to the first and third quartile break point, respectively. The bisector represents passive investment in the SPX.
Figure 6: Empirical CDF of out-of-sample investment returns of the active portfolio. This figure displays the ECDFs of the out-of-sample returns of the enhanced portfolios with 28-day options together with those of the corresponding passive strategy (black and gray points, respectively). Each graph represents one of the four scaling factors: $S = 1, 10, 100, 1000$. 

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