Strategic Risk Taking with Systemic Externalities

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Abstract

This paper studies strategic risk taking of highly-levered financial institutions within a structural model of credit risk. We consider a context in which systemic default induces externalities that amplify the private cost of financial distress. This represents a source of strategic interaction and mandates an analysis of financial institutions’ asset allocations in coalescence. We derive a unique strategic equilibrium in which two heterogenous institutions adopt polarized and stochastic risk exposures, without sacrificing full diversification. In the presence of systemic externalities, both financial firms are concerned with maintaining sufficient wealth in adverse states. To this purpose, the conservative institution reduces its risk exposure, whereas the aggressive institution optimally gambles on positive and negative outcomes by taking long and short positions in risky securities over time. This equilibrium mechanism increases the likelihood of a systemic crisis.

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1 Introduction

Over the past two decades, the financial system has evolved to be dominated by a small number of highly levered financial institutions.\footnote{In 2008, US commercial banks had leverage ratios (Total Assets/Total Equity) of roughly 15, while those of US investment banks were in the range of 20-30 (see \url{www.bis.org}).} Given the degree of interconnectedness between them and the sheer scale at which they operate, the decision making of each institution individually affects, and is affected by, the decisions of the others. Hence, their behavior should not be studied in isolation. This paper analyzes the risk taking of financial institutions in the presence of costly default and systemic externalities. We consider a context in which systemic default induces externalities that amplify the private cost of default. This represents a source of strategic interaction.

The recent financial crisis has highlighted the costliness of systemic default. When Lehman Brothers filed for Chapter 11 protection in September 2008, a few months after the collapse of Bear Sterns, not only were its assets subjected to an emergency liquidation at discount prices, it also set in motion what is described in the final report of the Financial Crisis Inquiry Commission as “[...] one of the largest, most complex, multi-faceted and far-reaching bankruptcy procedures ever filed in the United States. The costs of the bankruptcy administration are approaching one billion USD” (Financial Crisis Inquiry Commission, 2011). There is a vast literature detailing such direct costs of default, with an emphasis on the costs of legal settlement. The case of Lehman Brothers illustrates that when default is systemic, the increased complexity of the settlement process between different claimholders only amplifies these costs. The same applies for indirect costs of default, encompassing reputational damage, loss of trading opportunities and the liquidation of assets at fire-sales prices. The systemic nature of losses due to fire-sales is equally well-documented (Schleifer and Vishny, 1992; Acharya, Bharath and Srinivasan, 2007). However, the strategic behavior of financial institutions in the presence of such systemic externalities of default requires more understanding. Hence, we believe a comprehensive analysis of financial institutions’ strategic risk taking in a dynamic asset allocation framework is needed.

The challenge to better understand the strategic interactions at play, has recently been invigorated by policymakers’ efforts to design and reinforce macro-prudential regulation. By virtue of modeling the systemic externalities in a reduced form, our set-up easily extends to an analysis of macro-prudential policies. Several proposals have been circulated, ranging from Pigouvian taxes over systemic capital requirements and risk-surcharges to capital insurance (Acharya, Pedersen, Philippon and Richardson, 2010; Hart and Zingales, 2010; Hansen, Kayshap and Stein, 2011; Webber and Willison, 2011). All these initiatives share the objective of making the most global and systemically important institutions internalize the negative externalities they impose on the non-financial sector. While choosing among the proposed regulations is beyond the scope of this paper, we do contribute to the debate with a positive analysis of banks’ risk taking under incentives to
avoid a systemic crisis.

We consider two highly levered financial institutions (the banks) within a structural model of credit risk. As in Merton (1974), Black and Cox (1976) and Longstaff and Schwartz (1995), we take the capital structure of the two institutions as given, with debt already in place. To maintain as simple a setting as possible, and following Carlson and Lazrak (2010) among others, we assert that default may occur only upon maturity of the debt contracts, when the banks fail to repay the debt obligations. Each bank is run by a manager whose incentives are aligned with those of the equityholders. Both managers have access to a complete financial market and their task consists of selecting a portfolio of risky and riskless securities which endogenously determines the asset side of the banks’ balance sheet.

Default induces pecuniary costs affecting the banks’ budget constraint. Specifically, we adopt a cost function that is linear in the shortfall of the debt repayments, where the slope parameter identifies the cost per unit of default. In order to capture the presence of default externalities beyond the costs of idiosyncratic failure, we attribute a higher slope parameter to a systemic crisis, which is defined by the joint default of the institutions under consideration. In characterizing the banks’ optimal asset allocations, we appeal to the pure-strategy Nash equilibrium concept, in which each bank strategically accounts for the dynamic investment policies of the other bank, and the equilibrium policies of the two institutions are mutually consistent. As in Basak and Makarov (2011), by virtue of dynamically complete markets, the horizon equity is enough to characterize each bank’s strategy. This means that both managers optimally select an equity profile, which prescribes the value of their bank’s equity for any state of the world at maturity, given any possible equity profile of the other bank. Thus, we pin down the best response strategies.

For the case of heterogenous banks, we derive a unique equilibrium. The strategic interaction between financial institutions, captured by the interplay of their best response strategies, produces two distinct equilibrium equity profiles at maturity. The equilibrium has the following properties. First, in good states of the world both banks have optimal equity levels at least as high as their default boundaries, hence neither of them defaults. Second, in intermediate states wealth becomes expensive and only one bank can afford to maintain the level of the equity value higher or equal to the default boundary. This implies that intermediate states give rise to an idiosyncratic default regime. Finally, in the worst states of the world resisting default is too costly for both institutions, thus triggering a systemic crisis. Under these circumstances banks’ equilibrium equities are strictly below their respective default boundaries. For expositional convenience, we label the bank prone to idiosyncratic default as “early-defaulter” and the other one as “late-defaulter”.2

2Note that early and late refer to the state space dimension, not the time dimension, as default may only occur at maturity. In Section 3 we show that our results are qualitatively robust to varying the source of heterogeneity. Any unique equilibrium features an early- and a late-defaulter.
When banks are sufficiently homogenous, we show that multiple equilibria arise. This means that there are states of the world in which more than one pair of (mutually consistent) strategies can be part of an equilibrium. Specifically, this is the case when both banks agree on the fact that only one bank should default but they can not agree on which one. While selecting among the multiple equilibria is beyond the scope of this paper, we do establish the result that even in the extreme case of perfectly homogenous banks the equilibrium investment strategies are indeed heterogeneous. This confirms that in the case of the unique equilibrium, our results are effectively driven by the presence of the strategic interaction, and not by the ex-ante heterogeneity between the banks. We acknowledge that the existence of multiple equilibria creates scope for regulatory intervention.

When the equilibrium is unique, we provide a full characterization of the equilibrium investment policies under an isoelastic objective function and lognormal security prices. Our paper presents a tractable framework to study the dynamics of banks’ strategic exposure to risky securities. In particular, we focus on the level of each bank’s risk exposure (risk taking) and on their correlation (diversity) in the presence of strategic interactions. We evaluate our findings against a benchmark of no systemic default externalities on the one hand, and one of entirely costless default on the other hand.

While the banks’ efforts to internalize the systemic externalities induce them to implement less correlated asset allocations vis-à-vis the benchmarks (higher diversity), both banks attain full diversification by investing in the (same) mean-variance tangency portfolio. Thus, reduced correlation is not achieved by altering the composition of their portfolios, but rather by stochastically varying their exposure to the tangency portfolio. This means that at any point in time, conditional on the realization of a state of the world, the correlation between the banks’ portfolios is indeed equal to one. Unconditionally however, this correlation is less than one because the banks select optimal stochastic exposures that are not perfectly correlated. This set of results is at odds with the findings in Wagner (2011), where systemic liquidation costs lead agents to sacrifice diversification for diversity. In our model, full diversification is not compromised in the presence of negative systemic externalities because both banks can move along the efficient frontier by altering the fraction invested in the riskless bond. Furthermore, because of the stochastic nature of the banks’ investment policies, we ascertain that their correlation exhibits some interesting patterns. Diversity tends to increase upon the realization of intermediate states of the world, especially when time approaches maturity. These are precisely the circumstances under which banks are most concerned with systemic default and hence value diversity most.

Regarding the risk taking dynamics of the two financial institutions, we establish the novel result that strategic interaction drives a wedge in the equilibrium level of risk desired by the early- and the late-defaulter. While both banks aim at transferring horizon equity from good/intermediate states of the world to those characterized by (very costly) systemic default, in equilibrium they
choose polarized strategies. The late-defaulter adopts a conservative strategy. By implementing a low-risk investment policy, it generates sufficiently high wealth to finance an optimal equity profile where wealth is maintained above and at the default boundary in good and intermediate states, respectively. In contrast, the early-defaulter displays a more aggressive strategy. Close to maturity, in states of the world where both idiosyncratic and systemic default are likely to occur (intermediate states), the early-defaulter’s risk taking exhibits two radically opposite behaviors. Although defaulting at maturity is very likely, it either invests a high fraction of the assets in risky securities, or, at the other extreme, takes a short position in the tangency portfolio. Effectively, the only way the early-defaulter can allocate more wealth to systemic states at maturity is by accepting idiosyncratic default in intermediate states. Thus, in these states the bank allows its asset value to be very sensitive and positively correlated to economic fluctuations by investing heavily in the market for risky securities close to maturity. This high risk taking allows the bank to take away wealth from the states with idiosyncratic default and finance an asset value that is less sensitive to the severity of the systemic crisis. In the event of such a crisis, the early-defaulter also wants to increase the asset value in order to minimize the shortfall in the debt repayments. The only way it can deliver an asset profile that will jump upwards in case of joint default, is by investing in a portfolio that is negatively correlated with the economic fluctuations. This rationalizes the desire to short the market close to maturity.

Summarizing, we find that in the presence of systemic externalities, both banks are concerned with maintaining sufficient wealth in adverse states. However, while the conservative bank reduces its risk exposure, the radical bank optimally gambles on positive and negative outcomes, by taking either a large long or a short position in risky securities. We believe these results on banks’ strategic risk taking are new.

Given that the equilibrium equity profiles of the two financial institutions endogenously determine in which states idiosyncratic and systemic default occurs, they also carry implications for the probabilities of default. To appreciate the effect of negative systemic externalities (e.g., macro-prudential regulation) on the occurrence and the magnitude of systemic crises, we compare the equilibrium default probabilities and expected shortfalls in the strategic model to those in the benchmarks. Most notable, we find that in the presence of negative systemic externalities, both banks are more likely to default. This immediately implies that a systemic crisis becomes more likely.

This finding does not easily reconcile with the ex-ante objective of macro-prudential regulation to reduce the likelihood of a systemic event. We rationalize this outcome by recognizing that the systemic externalities implicitly generate what we term as a substitution and an income effect. The former captures the wealth transfer from idiosyncratic to systemic states which decreases the probability of a systemic crisis. The latter, instead, increases the probability of joint default since it reflects the asset value reduction caused by the negative externalities. The equilibrium outcome,
being the net of these two forces, shows that the income effect dominates for financial institutions with realistic leverage ratios. We complete this analysis by examining the expected shortfalls. We ascertain that, under extremely adverse economic conditions, the expected losses given default are lower in the presence of strategic interactions, by virtue of the banks’ wealth transfers into these states. In all, we believe these results are indicative of a friction between systemic default probabilities on the one hand and systemic losses on the other hand. Hence, this should not be overlooked in the design of a macro-prudential framework.

A last set of implications of our paper examines the pricing of credit spreads and credit default swap premiums for the two financial institutions. Since debt is a straightforward claim on the value of the assets, its equilibrium price is also endogenously determined by the strategic risk taking of the banks. Indeed, we find that the credit spreads inherit the distinctive features of banks’ risk exposures. This is especially clear for the early-defaulter whose radical risk taking behavior, translates into non-monotonic credit spreads. This means that lower credit spreads can be associated with worse states when time to maturity is low. More generally, we document a decrease in credit spreads for the most adverse states and an increase of the spreads for the more favorable ones. Once more, this reflects the banks’ wealth transfers from good to systemic states, in a bid to internalize the negative externalities associated with joint default. With respect to the CDS spreads, which are positively related to both the probability of default and the credit spread, we document that the price to pay for protection against default is higher in the strategic model than in the benchmarks. This is also true for the systemic states, where the credit spreads of the banks are markedly lower than for the benchmark models. Hence, we acknowledge the dominance of default probabilities on equilibrium CDS prices.

This paper relates to several strands of literature. We build on the literature of structural credit risk models. We employ the modeling approach presented by Basak and Shapiro (2005), allowing the asset-value dynamics to be endogenously determined. In this paper, we deviate from the standard structural contingent-claim approach, with exogenous asset-value dynamics, as first employed by Merton (1974) and extended, among others by Leland (1994), Longstaff and Schwarz (1995) and Anderson and Sundaresan (1996). By virtue of endogenizing the asset-values, this paper allows for an analysis of portfolio choice effects under strategic interactions. In this regard, our work is most closely related to Basak and Makarov (2011) who analyze dynamic portfolio strategies of money managers, in the presence of strategic interactions arising from relative performance concerns.

Notwithstanding a literature on banks’ portfolio choice in the presence of systemic externalities (Gorton and Huang, 2004; Wagner, 2011), our paper contributes by explicitly considering such externalities as a source of strategic interaction. This most clearly differentiates our paper from Wagner (2011). In a set-up with systemic liquidation costs, he considers atomistic agents’ investment in risky securities only and establishes that agents optimally sacrifice diversification for the sake of avoiding systemic failure. In our model, because both banks can move along the efficient
frontier by altering the fraction invested in the riskless bond, full diversification is not compromised.

In relating our paper to a more general literature on strategic behavior among banks, we acknowledge recent contributions by Perotti and Suarez (2002), Acharya (2009). Both works consider banks’ strategic portfolio decisions, in related yet different contexts of systemic default. While these papers consider potential gains from the acquisition of failed bank assets as a source of strategic interaction among financial institutions, we restrict our attention to amplified costs of joint default. With this, we touch upon the vast literature on costly default (Altman, 1984; Weiss, 1990; Andrade and Kaplan, 1998) and fire-sales (Schleifer and Vishny, 1992; Acharya, Bharath and Srinivasan, 2007). These papers extensively document how a systemic crisis exacerbates the costs of default, emphasizing the economic relevance of the systemic externalities considered in our model. A set of recent papers analyzes the optimal resolution of bank failures (Acharya and Yorulmazer, 2007, 2008; Farhi and Tirole, 2011). These articles show how systemic bail-outs distort the incentives for banks to correlate the risk in their investment choices. Our results on risk correlation in a context of negative systemic externalities complement their findings, by showing how banks’ preference for diversity evolves dynamically. Moreover, our full characterization of strategic risk taking adds a further layer of analysis to this literature.

Finally, and perhaps most importantly, we also add to the rapidly growing literature on systemic risk. Seminal contributions on the measurement of systemic risk include Adrian and Brunnermeier (2011) and Acharya, Pedersen, Philippon and Richardson (2010) among others. Much work in this field is motivated by the view that regulation should be designed in a way that financial institutions are penalized based on their contribution to systemic risk. By interpreting the systemic externalities in our model as any generic systemic policy, we are able to draw conclusions on financial institutions’ responsiveness to potential regulatory changes of this nature. We believe that our analysis of systemic crises, both in terms of likelihood and expected shortfall, within a workhorse dynamic asset allocation framework is new and delivers a rich set of implications regarding financial institutions’ strategic risk taking.

The remainder of the paper is organized as follows. Section 2 presents the economic set-up and lays out the micro-foundations for the strategic game in the presence of systemic externalities of default. Section 3 solves for the best-response strategies and characterizes the unique equilibrium for the case of heterogenous banks. Section 4 investigates the properties of the unique equilibrium. We analyze optimal risk taking, default probabilities and shortfalls, and debt pricing. Section 5 concludes. Proofs and minor results are derived in the Appendix.
2 The Model

2.1 The Economic Setting

We consider a continuous-time, finite horizon economy, \( t \in [0, T] \), in which the uncertainty is represented by a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) on which an \( N \)-dimensional standard Brownian motion, \( w_t = (w_{1t}, \cdots, w_{Nt})'\), is defined. We assume that all stochastic processes are adapted to the augmented filtration \( \{\mathcal{F}_t\} \) generated by \( w \), and that regularity conditions that make the processes well-defined (Karatzas and Shreve, 1998) are satisfied.

Financial market. Financial investment opportunities are given by \( N + 1 \) assets: an instantaneously riskless bond and \( N \) risky securities, whose prices evolve according to the following dynamics

\[
\begin{align*}
    dB_t &= B_trdt \\
    dS_t &= S_t\mu dt + S_t\sigma dw_t.
\end{align*}
\]

The bond provides a continuously compounded constant interest rate of \( r \), whereas \( \mu \) and \( \sigma \) represent the \( N \)-dimensional vector of mean returns and the \( N \times N \) non-degenerate volatility matrix of the risky securities, respectively. Markets are dynamically complete, implying the existence of a unique state price density process \( \xi \) such that

\[
    d\xi_t = -\xi_trdt - \xi_t\kappa'dw_t
\]

where \( \kappa \equiv \sigma^{-1}[\mu - r1] \) is the \( N \)-dimensional market price of risk, \( 1 \) is the \( N \)-dimensional vector \((1, \cdots, 1)'\), and \( \xi_0 \) is set to 1.

Agents. Our economy is populated by two financial institutions, which for simplicity we will refer to as banks hereafter. If we let \( V_{it} \) denote the value of the assets of bank \( i \) at time \( t \), and \( W_{it} \) and \( D_{it} \) the value of the equity and of the debt, respectively, then the following accounting identity must hold,

\[
    V_{it} = W_{it} + D_{it}.
\]

Each bank \( i \in \{1, 2\} \) is run by a manager whose incentives are aligned with equityholders’ interests. The manager is guided by an isoelastic objective function defined over the terminal value of the
equity (the bank’s net wealth),

\[ u_i(W_{iT}) = \frac{W_{iT}^{1-\gamma_i} - 1}{1 - \gamma_i}, \quad \gamma_i > 0. \]  

(5)

It is important to stress that the interpretation of the aforementioned objective function is broader than a standard concave utility function. Indeed, it could capture managerial self interest, compensation structures, concave non-stochastic investment opportunities beyond the terminal date, and, more in general, market frictions (Allen and Santomero, 1998; John and John, 1993; John, Saunders and Senbet, 2000; Froot, Scharfstein and Stein, 1993; Froot and Stein, 1998).

The manager of bank \( i \) maximizes the expected value of (5) by dynamically choosing an investment policy \( \pi_{it} \), which denotes the \((N\text{-dimensional})\) vector of fractions of bank \( i \)'s assets invested in each risky security, given an initial capital of \( W_{i0} \). We refer to \( \pi_{it} \) as the risk taking of bank \( i \). Clearly, \((1 - \pi_{it}'1)\) pins down the fraction of bank \( i \)'s assets invested in the riskless bond. Hence, the optimization problem faced by the manager of bank \( i \) is subject to the dynamics of the value of the assets:

\[ dV_{it} = V_{it}[r + \pi_{it}'(\mu - r1)]dt + V_{it}\pi_{it}'\sigma dw_t. \]  

(6)

2.2 Leverage, Default and Externalities

As in many structural models of credit risk (Merton, 1974; Leland, 1994; Longstaff and Schwarz, 1995), we do not derive the optimal capital structure of the two financial institutions, but rather we take it as given and study the implications of their strategic interaction on optimal investment decisions and systemic risk.\(^ 3\) Therefore, we assume that both banks are levered; specifically, they are bound by a zero-coupon debt contract with face value of \( F_i \) and price \( D_{i0} \) at the initial date, where the latter will be determined endogenously by the optimal asset choice of bank \( i \). The presence of debt in the banks' balance sheets captures the possibility of these institutions defaulting.

Debt contract. Financial distress may occur at the terminal date \( T \) and it is triggered if bank \( i \) fails to repay its debt obligation, equal to the face value \( F_i \). In such a case, debtholders can force liquidation or reorganization and can seize only a fraction of the bank’s total assets, \((1 - \beta_i) V_{iT} \), while the remaining \( \beta_i V_{iT} \) is retained by the equityholders. By assuming \( \beta_i > 0 \), we capture violations of the Absolute Priority Rule (APR), a fact which is extensively documented in the empirical literature (Franks and Torous, 1989, 1994; Eberhart, Moore, and Roenfeldt, 1990; Weiss, 1990; Betker, 1995). Departures from APR can be rationalized as the optimal outcome

\(^3\)Optimal capital structure within a credit risk model is studied in Leland (1994), Leland and Toft (1996). However, these papers assume exogenous asset value processes.
of a bargaining game among corporate claimants after time $T$ (Anderson and Sundaresan, 1996; Mella-Barral and Perraudin, 1997, Fan Sundaresan, 2000; Acharya, Huang, Subrahmanyam and Sundaram, 2006; Garlappi, Shu and Yan, 2008).\footnote{As a complementary interpretation, we can view $\beta_i V_{iT}$ as bank $i$’s intangible assets, which can not be collateralized and hence transferred to the debtholders.} Hence, the payoff of the debt contract is given by:

$$D_{iT} = \min\{(1 - \beta_i) V_{iT}, F_i\}$$  \hspace{1cm} (7)

where $\beta_i \in [0, 1]$. Bank $i$ enters financial distress when $V_{iT} < F_i/(1 - \beta_i)$. It is worth clarifying that in this model there is no formal distinction between default and distress: what really matters is that in both cases the bank can not repay the face value of the debt. In what follows, we make use of both terms interchangeably.

**Cost of default and systemic externalities.** Default is costly. An extensive literature has documented the nature and severity of (direct and indirect) costs of financial distress. Guided by these insights, and following Basak and Shapiro (2005), we model cost of default by adopting the following reduced form:

$$C_{iT} = \begin{cases} 
0 & \text{if } D_{iT} = F_i \\
\phi + \lambda (F_i - D_{iT}) & \text{if } D_{iT} < F_i \land D_{jT} = F_j \\
\phi + (\lambda + \eta_i) (F_i - D_{iT}) & \text{if } D_{iT} < F_i \land D_{jT} < F_j,
\end{cases}$$  \hspace{1cm} (8)

for any $i \in \{1, 2\}$ and $j \neq i$. Upon default, $D_{iT} < F_i$, bank $i$ incurs a fixed and a proportional costs, $(\phi, \lambda) \geq 0$, where the latter is proportional to the extent of default $(F_i - D_{iT})$. When financial distress is systemic, that is when both banks default on their debt, the proportional cost become equal to $(\lambda + \eta_i)$. If $\eta_i > 0$ we have a negative systemic externality; if $-\lambda < \eta_i < 0$ we have a positive systemic externality. Hence, systemic defaults can be more or less costly than idiosyncratic defaults depending on whether the systemic externality is negative or positive. We allow the systemic components of the cost to be heterogeneous among banks.

Besides the advantage of tractability, this parsimonious yet flexible specification allows us to capture, within the same theoretical framework, different economic scenarios:

- **Direct financial distress costs.** Direct costs of default encompass the costs of lawyers, accountants, and other professionals involved in the bankruptcy filling, including the value of managerial time spent to this purpose (Warner, 1977; Weiss, 1990; Andrade and Kaplan, 1998). These costs are both fixed and proportional, and it is reasonable to assume that in the event of a systemic crisis they would rise because of the increased complexity of negotiating the disputes between claimholders.
• **Indirect financial distress costs.** Impaired business reputation, loss of trading opportunities (Dow and Rossienisky, 2001), loss of market share (Opler and Titman, 1994), liquidation of assets at fire-sales prices (Allen and Gale, 1994, 1998; Acharya and Yorulmazer, 2007; Wagner, 2011) represent the large part of indirect costs of default. Since such opportunity costs depend on the market setting, they become more severe during a systemic crisis. For instance, it is more likely that some of the assets of the distressed institutions are acquired by investors who are not the most efficient user of these assets, thus valuing them below their fundamental value (Williamson, 1988; Shleifer and Vishny, 1992).

• **Macro-prudential/systemic regulation.** Inspired by the events of the recent financial crisis, several proposals to contain systemic risk have been advocated (Acharya, Pedersen, Philippon and Richardson, 2010; Hart and Zingales, 2010; Hansen, Kayshap and Stein, 2011; Webber and Willison, 2011). A general consensus seems to highlight the inability of micro-prudential regulations to prevent the collapse of the financial system as a whole, and the urgent need for macro-prudential regulations. Of these kind, among others, are Pigouvian taxes, systemic capital requirements, systemic-based risk constraints, capital insurance, systemic risk surcharges, all tailored to induce systemic financial institutions to internalize their externalities on the entire economy. In other words, they are designed to induce the financial sector to bear some of the social costs that they would generate in the event of a crisis. Discussing different aspects of the Dodd-Frank Act, Acharya, Cooley, Richardson and Walter (2010) concisely summarizes this view:

> “The basic idea is that, to the extent these stricter standards impose costs on financial firms, these firms will have an incentive to avoid them and therefore be less systemically risky.”

Therefore, we can deem our postulated cost function as capturing (in reduced form) the implementation of such systemic regulations through the (positive) coefficient $\eta_i$.\(^5\)

• **Systemic bail-out.** While the previous scenarios represent negative systemic externalities, our framework can also accommodate positive systemic externalities such as explicit and implicit promises and transfers from the rest of the economy to the financial sector. Moral hazard problems related to the resolution of bank failures (Acharya and Yorulmazer, 2007, 10).

\(^5\)As an example, consider the optimal (Pigouvian) tax system obtained in Acharya, Pedersen, Philippon and Richardson (2010); each financial institution is taxed based on: (i) the expected shortfall in case of default, and (ii) the expected shortfall in case of a systemic crisis. Therefore, the expected default cost that the banks face in our model can be easily interpreted as a systemic tax:

$$
E[\xi_tC_T] = \lambda E[\xi_T(F_i - D_{i,T})|single \ distress]P(single \ distress) + 
\eta_i E[\xi_T(F_i - D_{i,T})|systemic \ distress]P(systemic \ distress) + K
$$

where the event $\{systemic \ distress\} \subset \{single \ distress\}$ and $K$ captures the expected fixed costs which do not depend on the banks’ shortfall.
2008; Panageas, 2010; Farhi and Tirole, 2011) can make the cost of distress in the event of a financial crisis lower than the one in the event of a single default. In our model, this corresponds to negative values of $\eta_i$.

Therefore, the coefficients in our cost specification are meant to capture the net effect of all the potential determinants of the aforementioned distress costs. Note that when $\eta_i$ is equal to zero, systemic externalities are absent and the two banks are not connected with each other, thus behaving as the single borrower in Basak and Shapiro (2005). This implies that their optimal investment problems can be solved independently. In contrast, when externalities are present, bank $i$ acts strategically by taking into account the effect that bank $j$ has on bank $i$’s cost function, knowing that bank $j$ takes into account the effect that bank $i$ has on bank $j$’s cost function, and so on. Hence, the optimal investment policies of the two banks are the equilibrium outcome of a strategic game.

2.3 Default Regions

Before formally defining the strategic game between the two financial institutions, let us determine the banks’ equity and total asset value in all the possible regions of default. At the terminal date $T$, taking into account of the default cost, the accounting identity in (4) becomes equal to

$$V_{iT} - C_{iT} = W_{iT} + D_{iT}$$

so that the value of the equity of bank $i$ at time $T$ is given generically by

$$W_{iT} = V_{iT} - \min\{(1 - \beta_i)V_{iT}, F_i\} - \left(\phi + \lambda(F_i - \min\{(1 - \beta_i)V_{iT}, F_i\}) + \eta_i(F_i - \min\{(1 - \beta_i)V_{iT}, F_i\})I_{\{(1 - \beta_i)V_{iT} < F_i\}}\right)I_{\{(1 - \beta_i)V_{iT} < F_i\}}.$$  

Based on this we can consider the following sub-cases:

**No-default region.** None of the two banks are in financial distress: $(1 - \beta_1)V_{1T} \geq F_1$ and $(1 - \beta_2)V_{2T} \geq F_2$.

$$W_{iT} = V_{iT} - F_i \Rightarrow V_{iT} = W_{iT} + F_i$$  

for $i \in \{1, 2\}$. 

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Single default region. Only bank $i$ is in financial distress: $(1 - \beta_i)V_{iT} < F_i$ and $(1 - \beta_j)V_{jT} \geq F_j$.

\[ W_{iT} = V_{iT}[\beta_i + \lambda_i(1 - \beta_i)] - [\phi + \lambda F_i] \quad \Rightarrow \quad V_{iT} = \frac{W_{iT} + [\phi + \lambda F_i]}{\beta_i + \lambda(1 - \beta_i)} \] (12)

\[ W_{jT} = V_{jT} - F_j \quad \Rightarrow \quad V_{jT} = W_{jT} + F_j \] (13)

Systemic default region. Both banks are in financial distress: $(1 - \beta_1)V_{1T} < F_1$ and $(1 - \beta_2)V_{2T} < F_2$.

\[ W_{iT} = V_{iT}[\beta_i + (\lambda + \eta_i)(1 - \beta_i)] - [\phi + (\lambda + \eta_i)F_i] \quad \Rightarrow \quad V_{iT} = \frac{W_{iT} + [\phi + (\lambda + \eta_i)F_i]}{\beta_i + (\lambda + \eta_i)(1 - \beta_i)} \] (14)

for $i \in \{1, 2\}$.

In order to express the default boundary in terms of the value of the equity, let $W_i$ denote the lower-bound value of $W_{iT}$ in the no-default region:

\[ W_i : \quad (1 - \beta_i)(W_i + F_i) = F_i \quad \Rightarrow \quad W_i \equiv \frac{\beta_i F_i}{1 - \beta_i}, \] (15)

for $i \in \{1, 2\}$. When bank $i$’s equity at maturity is greater than or equal to the threshold $W_i$, the bank does not default. Note, however, that the upper-bound value of $W_{iT}$ in the default regions is given by $W_i - \phi$. Therefore, the fixed cost creates a discontinuity such that the value of the equity at maturity can not take values in the interval $[W_i - \phi, W_i]$. In the next section, we verify that the optimal policies of the two banks satisfy this condition. To avoid abuse of notation, throughout this paper we refer to the event $\{W_{iT} < W_i\}$ as default. Since there is a one-to-one mapping between equity and asset values in all the default regions, we can express all the relevant quantities as a function of the horizon equities ($W_{iT}, W_{jT}$). In the following Lemma we do this for the cost function.

**Lemma 1.** Bank $i$’s cost of default can be written as

\[
C_{iT}(W_{iT}, W_{jT}) = \begin{cases} 
0 & \text{if } W_{iT} \geq W_i \\
x_i \phi + (1 - x_i)(W_i - W_{iT}) & \text{if } W_{iT} < W_i \land W_{jT} \geq W_j \\
z_i \phi + (1 - z_i)(W_i - W_{iT}) & \text{if } W_{iT} < W_i \land W_{jT} < W_j,
\end{cases}
\] (16)

for $j \neq i$, where

\[
\begin{align*}
x_i &\equiv \frac{1}{1 + \hat{\lambda}_i}, & z_i &\equiv \frac{1}{1 + (\hat{\lambda}_i + \hat{\eta}_i)}, & \hat{\lambda}_i &\equiv \lambda \left(\frac{1 - \beta_i}{\beta_i}\right), & \hat{\eta}_i &\equiv \eta \left(\frac{1 - \beta_i}{\beta_i}\right).
\end{align*}
\] (17)

If the systemic externality is negative (positive), then $x_i > z_i$ ($x_i < z_i$).

*Proof.* See the Appendix.
2.4 Martingale Representation and The Strategic Game

The manager of bank $i$ maximizes the expected value of the objective function over the value of the final horizon equity, subject to the dynamic budget constraint in (6), and the default cost in (8). Under the assumption of dynamically complete markets, we can solve the dynamic optimization problem of bank $i$ using the martingale representation approach (Karatzas, Lehoczky and Shreve, 1987; Cox and Huang, 1989). This entails solving the following static problem:

$$\max_{W_{iT}} \mathbb{E}[u_i(W_{iT})] \quad s.t. \quad \mathbb{E}[\xi T V_{iT}] \leq V_{i0}$$

where $V_{i0} = W_{i0} + D_{i0}$ and $V_{iT} = W_{iT} + D_{iT} + C_{iT}$. Since markets are dynamically complete, the debt contract is fairly priced, $D_{i0} = \mathbb{E}[\xi T D_{iT}]$, and the bank’s problem can be restated as

$$\max_{W_{iT}} \mathbb{E}[u_i(W_{iT})] \quad s.t. \quad \mathbb{E}[\xi T (W_{iT} + C_{iT})] \leq W_{i0}$$

where, by Lemma 1, $C_{iT} = C_{iT}(W_{iT}, W_{jT})$.

The two banks are interconnected through the (systemic) cost of default: the choice of one bank to default affects and is affected by the choice of the other. Hence, they play a strategic dynamic game. We solve this game for an equilibrium in time-dependent strategies, where the strategy of each bank consists of a non-negative horizon equity $W_{iT}$ and a sequence of fractions of bank’s assets to invest in the risky financial securities $\{\pi_{it}\}_{t \in [0,T]}$. By virtue of dynamically complete markets, (i) the horizon equity $W_{iT}$ is enough to characterize each bank’s strategy, and (ii) the dynamic game can be viewed as a game played only at time 0, where each bank decides its strategic actions for the whole time period $[0,T]$. This is equivalent to solving for an open-loop equilibrium.\(^6\) The two institutions play such game under the assumption of complete information, given that the set $\{W_{i0}, F_i, u_i(\cdot), \beta_i, \phi, \lambda, \eta_i\}$, for any $i \in \{1, 2\}$, is common knowledge.

**The game.** Let $\langle \mathcal{I}, \Omega, \mathbb{P}, (\mathcal{S}_i), (v_i) \rangle$ denote the strategic game, where $\mathcal{I}$ refers to the set of players; $\mathbb{P}$ denotes a probability measure defined over the set of states $\Omega$; $\mathcal{S}_i$ represents the nonempty set of strategies $s_i$ available to bank $i$, and $v_i$ its payoff function. Specifically,

- $\mathcal{I} = \{1, 2\}$, $\Omega = \mathbb{R}^{++}$, $\mathbb{P} : \log(\xi T) = - (r + \kappa^2/2) T - \kappa w T$;
- $\mathcal{S}_i = \{W_{iT}(\xi T) : \Omega \rightarrow \mathbb{R}^{++} : \mathbb{E}[\xi T (W_{iT} + C_{iT})] \leq W_{i0}\}$
- $v_i : (\mathcal{S}_i \times \mathcal{S}_j) \rightarrow \mathbb{R}$ is such that $v_i(W_{iT}, W_{jT}) = \max_{W_{iT} \in \mathcal{S}_i} \mathbb{E}[u_i(W_{iT})]$

\(^6\)We refer to Basar and Olsder (1982, 1995) and Dockner, Jorgensen, Van Long and Sorger (2000) for an exhaustive discussion on open- and closed-loop equilibria and on equilibrium concepts in differential games. Carlin, Lobo, and Viswanathan (2007) and Oehmke (2012) consider open-loop equilibria of dynamic trading games in the presence of liquidity frictions. Note that, as highlighted by Back and Paulsen (2009), there are issues in defining closed-loop equilibria for dynamic games in continuous-time and continuous-action; for this reason we leave the investigation of such equilibria for future research.

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**Definition 1 (Best Response Strategies).** Consider the strategic game \((\mathcal{I}, \Omega, \mathbb{P}, (\mathcal{S}_i), (v_i))\). For any \(W_{jT}(\xi_T) \in \mathcal{S}_j\), let \(B_i(W_{jT})\) define the set of bank i’s best response strategies given \(W_{jT}\):

\[
B_i(W_{jT}) = \{ W_{iT}(\xi_T) \in \mathcal{S}_i : v_i(W_{iT}, W_{jT}) > v_i(W'_{iT}, W_{jT}) \text{ for all } W'_{iT}(\xi_T) \in \mathcal{S}_i \}.
\]

Then, \(\hat{W}_{iT}(W_{jT})\) denotes an element of \(B_i(W_{jT})\).

**Definition 2 (Pure-Strategy Nash Equilibrium).** A pure-strategy Nash equilibrium of the strategic game \((\mathcal{I}, \Omega, \mathbb{P}, (\mathcal{S}_i), (v_i))\) is a profile of strategies \((W^*_iT, W^*_jT) \in (\mathcal{S}_i, \mathcal{S}_j)\) for which

\[
W^*_iT \in B_i(W^*_jT) \text{ for all } i \in \mathcal{I}.
\]

## 3 Strategic Equilibrium with Negative Systemic Externalities

In the current section we solve for the strategic game played by the two levered financial institutions subject to default costs, as described in the earlier section. First we derive each bank’s best response strategy, then we characterize the strategic equilibrium by selecting those strategies that are mutual consistent.

### 3.1 Best Response Strategies

The manager of bank \(i\) faces the optimization problem in (19): he maximizes the expected value of the objective function defined over the equity value at the terminal date, subject to the static budget constraint. The budget constraint states that that expected discounted value of the sum of the horizon equity plus potential default costs (discounted with the pricing kernel of the economy, \(\xi_T\)) can not be higher than the value of the initial capital. In simpler terms, the implemented policy must be affordable.

As equation (16) highlights, the cost of default of bank \(i\) is affected by the policy adopted by bank \(j\). Depending on whether bank \(j\) is solvent or not at maturity, bank \(i\)’s cost function would exhibit low or high proportional costs, respectively. The best response strategy of bank \(i\) prescribes the optimal level of equity at time \(T\) for any possible realization of the uncertainty, \(W_{jT}(\xi_T)\), conditional on the strategy of bank \(j\). In other words, bank \(i\) takes \(W_{jT}\) as given and solves the optimization problem in (19) for all possible values of \(W_{jT}\). However, since the cost function of bank \(i\) is not affected by the level of \(W_{jT}\) per se, but rather by whether \(W_{jT}\) is above or below the default boundary \(\underline{W}_j\), the bank needs to solve only two optimization problems. Each
problem is conditional to one of the two collectively exhaustive and mutually exclusive events,

\[ \{W_{jT} \geq W_j\} \quad \text{and} \quad \{W_{jT} < W_j\}. \]

Because of the nonlinearity and discontinuity in the default cost functions, the banks’ optimization problems are non-standard as they are not globally concave. In fact, they exhibit local convexity around the default boundaries \( W_i \), for \( i \in \{1, 2\} \). To tackle this issue, we adapt the common convex-duality approach (e.g., Karatzas and Shreve, 1998) to incorporate kinks and discontinuities in the objective and in the budget constraint.\(^7\) In the following Proposition we characterize the banks’ best response strategies explicitly in closed-form.

**Proposition 1.** The best response function of bank \( i \) with respect to bank \( j \), for any \( i \neq j \), is given by

\[
\hat{W}_{iT}(W_{jT}) = \begin{cases} 
(y_i \xi_T)^{\frac{1}{\gamma_i}} & \text{if } \xi_T \leq \xi_i \\
W_i & \text{if } \xi_i < \xi_T \leq \bar{\xi}_i \\
(y_i x_i \xi_T)^{\frac{1}{\gamma_i}} + \left[ W_i - (y_i x_i \xi_T)^{\frac{1}{\gamma_i}} \right] \mathbb{1}\{W_{jT} < W_i\} & \text{if } \bar{\xi}_i < \xi_T \leq \bar{\bar{\xi}}_i \\
(y_i x_i \xi_T)^{\frac{1}{\gamma_i}} + \left[ (y_i z_i \xi_T)^{\frac{1}{\gamma_i}} - (y_i x_i \xi_T)^{\frac{1}{\gamma_i}} \right] \mathbb{1}\{W_{jT} < W_i\} & \text{if } \xi_T > \bar{\bar{\xi}}_i.
\end{cases}
\]

The thresholds \((\xi_i, \bar{\xi}_i, \bar{\bar{\xi}}_i)\) are given by

\[
\xi_i \equiv \frac{(W_i)^{-\gamma_i}}{y_i}, \quad \bar{\xi}_i \equiv \frac{\alpha_i}{y_i x_i}, \quad \bar{\bar{\xi}}_i \equiv \frac{\alpha_i}{y_i z_i},
\]

where \( \alpha_i > (W_i)^{-\gamma_i} \) is the solution to the following equation

\[
W_i^{1-\gamma_i} - \gamma_i \alpha_i^{1-\frac{1}{\gamma_i}} - \alpha_i (W_i - \phi) (1 - \gamma_i) = 0,
\]

The Lagrange multiplier \( y_i \) is set such that

\[
\mathbb{E}\left[ \xi_T \hat{W}_{iT}(W_{jT}, \xi_T; y_i) + \xi_T C_{iT}(\hat{W}_{iT}(W_{jT}, \xi_T; y_i)) \right] - W_{i0} = 0.
\]

If the systemic externality is negative, \( \eta_i > 0 \), then the following ordering holds: \( \xi_i < \bar{\xi}_i < \bar{\bar{\xi}}_i \).

**Proof.** See the Appendix.

\(^7\)Examples of non-standard optimization problems include Carpenter (2000); Basak and Shapiro (2001, 2005); Basak, Pavlova and Shapiro (2007); Carlson and Lazrak (2010); Basak and Makarov (2011).
the best response behavior adopted in each of these regions, recall that the objective of the manager of the bank is to choose the optimal equity profile to implement at maturity. Such profile, which must be affordable, prescribes the how much net wealth (equity) to have at maturity for all possible economic scenarios.

For a graphical representation of the best response strategy, Panel A in Figure 1 plots the optimal equity profile of bank $i$ at time $T$ as a function of the realizations of the economic uncertainty (the state price density $\xi_T$). In Panel A1 the equity profile is conditional on bank $j$ not defaulting; in Panel A2 it is conditional on bank $j$ being in distress; the graph in Panel A3 combines the previous two. Panel B, on the contrary, provides a more commonly used representation of the best response strategy by plotting the optimal equity of bank $i$ at time $T$ as a function of the equity of bank $j$. Panel B1-B4 correspond to different economic scenarios.
The economic intuition underlying the optimal equity profile is as follows. When economic conditions are good ($\xi_T \leq \xi_i$), the price (per unit of probability $\mathbb{P}$) of one unit of wealth (that is $\xi_T$) is low. Since wealth in these states of the world is “cheap” it is optimal for the manager of bank $i$ to have high equity. High equity means that the firms do not default. Moreover, this holds whether or not bank $j$ is in distress. When economic prospects deteriorate ($\xi_i < \xi_T \leq \bar{\xi}_i$), the price of wealth become more expensive and bank $i$ would default if there were no costs associated to this choice. However, to avoid distress costs, it is optimal to “buy” the minimum amount of wealth that allows the bank to be solvent. This corresponds to an equity value equal to the (constant) default boundary $W_i$, as shown by the black flat line in Panel A3. Even in this case the financial soundness of bank $j$ does not affect the decision of bank $i$ to resist default.

If economic conditions get worse ($\bar{\xi}_i < \xi_T \leq \bar{\bar{\xi}}_i$), then the trade-off between paying default costs if defaulting and paying expensive wealth (at the expenses of wealth in other states of the world) if resisting default becomes dependent on the equity choice of bank $j$. Indeed, if bank $j$ does not default, it is optimal for bank $i$ to do so because distress costs (per unit of default) are, in relative terms, low. In contrast, if bank $j$ is in distress, it is optimal for bank $i$ to resist default in order to avoid high systemic costs. Finally, in the very bad states of the world ($\xi_T > \bar{\bar{\xi}}_i$), the price of wealth is so high that makes default the optimal choice for bank $i$. Note that, in these states, the financial condition of bank $j$ does not affect the bank $i$’s decision to default but rather the level of default. In fact, if bank $j$ is in distress, the loss given default of bank $i$ is lower, thus implying a higher equity value. This is explained by the attempt of bank $i$ to at least reduce the unavoidable systemic costs, as captured by the gap between the red and the black lines in Panel A3, or by the jump in correspondence to $W_j$ in Panel B4.

Note that in an economic environment without externalities ($\eta_i = 0$, in our model), the optimal policy of bank $i$ would coincide with its best response strategy since conditioning on any policy adopted by bank $j$ has no impact on bank $i$’s decision. However, in a strategic setting ($\eta_i > 0$), the optimal policies of the two institutions are given by the interaction of their best response strategies. Such “interaction” pins down those strategies that are mutually consistent with each other, and it is formally characterized in the next section.

### 3.2 Equilibrium Strategies

In the current section, we show that the strategic interaction between financial institutions can generate both unique and multiple equilibria. We establish the conditions for these two possible outcomes, and we provide the economic intuition of the underlying mechanism.

Since each bank’s best response strategy is characterized by three different thresholds, the entire state space can be generically divided into seven partitions. Definition 2 implies that an
equilibrium of the strategic game exists provided that: (i) a Nash equilibrium exists for any \( \xi_T \) in the seven partitions; (ii) each bank’s budget constraint is satisfied. In particular, following the characterization in Basak and Makarov (2011), a Nash equilibrium is unique if for any \( \xi_T \) there is one and only one strategy-pair \( (W^*_1, W^*_2) \) both banks agree on and have no incentive to deviate from:

\[
W^*_1 = \hat{W}^*_1(W^*_2, \xi_T) \quad \text{and} \quad W^*_2 = \hat{W}^*_2(W^*_1, \xi_T) \quad \forall \xi_T.
\]

Multiple equilibria, instead, occur if for each \( \xi_T \) the banks agree on at least one strategy-pair \( (W^*_1, W^*_2) \) and for some states \( \xi_T \) they agree on more than one. The following Proposition provides the condition for uniqueness of a pure-strategy equilibrium, and, in such case, characterizes the banks’ equilibrium strategies.

**Proposition 2.** Consider two heterogeneous banks. If the initial capital of one, and only one, of the two banks is below some threshold, \( W_{j0} \leq \hat{W}_{j0} \), then the Nash Equilibrium is unique, otherwise multiple equilibria occur. When unique, the equilibrium is characterized by

\[
W^*_iT = \begin{cases} 
(y^*_i \xi_T)^{-\frac{1}{\gamma_i}} & \text{if } \xi_T \leq \xi^*_i \\
W_i & \text{if } \xi^*_i < \xi_T \leq \Xi^*_i \\
(y^*_i z_i \xi_T)^{-\frac{1}{\gamma_i}} & \text{if } \xi_T > \Xi^*_i,
\end{cases}
\]

\[
W^*_jT = \begin{cases} 
(y^*_j \xi_T)^{-\frac{1}{\gamma_j}} & \text{if } \xi_T \leq \Xi^*_j \\
W_j & \text{if } \Xi^*_j < \xi_T \leq \Xi^*_i \\
(y^*_j x_j \xi_T)^{-\frac{1}{\gamma_j}} & \text{if } \Xi^*_j < \xi_T \leq \Xi^*_i \\
(y^*_j z_j \xi_T)^{-\frac{1}{\gamma_j}} & \text{if } \xi_T > \Xi^*_i.
\end{cases}
\]

(26)

where \( y^*_i \) and \( y^*_j \) are such that

\[
E[\xi_T(W^*_iT(y^*_i)) + C^*_iT(W^*_iT(y^*_i)))] = W_{i0}
\]

(27)

\[
E[\xi_T(W^*_jT(y^*_j, y^*_i)) + C^*_jT(W^*_jT(y^*_j, y^*_i)))] = W_{j0}.
\]

(28)

The thresholds \( \hat{W}_{j0} \) for \( j \in \{1, 2\} \) are defined in the Appendix.

**Proof.** See the Appendix.

**Remark 1.** To highlight the generality of our result, the equilibrium is solved in the Appendix for a generic objective function \( u_i(\cdot) \) that satisfies the usual assumptions: it is strictly increasing, strictly concave, twice continuously differentiable, and it satisfies the Inada conditions.

Proposition 2 reveals that, if there is some degree of heterogeneity among banks, the equilibrium of the strategic game is unique. The intuition is that sufficient heterogeneity guarantees that in equilibrium there are no states of the world (at time \( T \)) in which both banks want to default but only one can. This would be the case when the two regions

\[
\xi_1 < \xi_T \leq \bar{\xi}_1 \quad \text{and} \quad \xi_2 < \xi_T \leq \bar{\xi}_2
\]

(29)

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overlap. The existence of such set of states,

\[ \xi_i < \xi_T \leq \xi_j, \]  

creates multiple equilibria since there is no rule/mechanism to select which bank should default for each state in that set. The banks agree on the fact that only one bank should default but they can not agree on which one. For instance, suppose that \( \xi_1 < \xi_T \leq \xi_2 \). According to (22), bank 1 wants to default if bank 2 is solvent, and resist default otherwise. At the same time, bank 2 wants to default only if bank 1 is solvent. Therefore, for any state of the world in that interval, two possible strategy-pairs can be part of an equilibrium:

\[ (W_{1T}^* < W_1, W_{2T}^* = W_2) \quad \text{or} \quad (W_{1T}^* = W_1, W_{2T}^* < W_2). \]  

(31)

We can conclude that the condition for uniqueness requires that the two regions defined in (29) do not overlap:

\[ \bar{\xi}_2 < \bar{\xi}_1 \quad \text{or} \quad \bar{\xi}_1 < \bar{\xi}_2. \]  

(32)

In the Appendix, we show that (32) translates into a simple threshold condition on the initial capital of each of two banks, \( W_{i0} \leq \bar{W}_{i0} \) for \( i \in \{1, 2\} \). When one, and at most one, of these conditions is satisfied, the equilibrium is unique. While selecting among multiple equilibria is beyond the scope of this paper, we do establish the result that also homogenous banks adopt heterogenous investment strategies, in an attempt to minimize the systemic externality.8

**Corollary 1.** Consider two homogeneous banks. Then:

(i) There are Multiple Nash Equilibria.

(ii) A symmetric equilibrium, \( W_{1T}^* = W_{2T}^* \), does not exists.

Proof. See the Appendix. \( \Box \)

Let us now focus on the unique equilibrium. Proposition 2 unveils the distinct equilibrium strategies of the two institutions. In particular, the unique equilibrium is such that the two banks can be classified based on their default behavior. If \( W_{j0} \leq W_{j0} \), we label bank \( j \) as early-defaulter, and bank \( i \) as late-defaulter, because in equilibrium the former will default “earlier” in the state-space dimension (and not in the time-dimension since default can happen only at the final horizon \( T \)). Indeed, the early-defaulter bank will enter financial distress if \( \xi_T > \bar{\xi}_j \), whereas the late-defaulter if \( \xi_T > \bar{\xi}_i \). Given (32), it is straightforward to show that \( \bar{\xi}_j < \bar{\xi}_i \).

---

8Appendix C presents an example of multiple equilibria with homogeneous banks.
We now describe the unique equilibrium strategies. Without loss of generality, let bank 1 be the late-defaulter. Equation (26) highlights how in equilibrium bank 1 can afford resisting default in a larger set of states, and hence its equity profile exhibits a wider region in which the optimal equity levels at the default boundary $W_1$. When extremely poor economic conditions materialize ($\xi_T > \bar{\xi}_1$), default occurs. Despite the discontinuity at $\bar{\xi}_1$ caused by the fixed cost of default, the equity profile of bank 1 is monotonic in the state price density.

In contrast, bank 2, the early-defaulter, finds it very expensive to maintain its equity value to a level ($W_2$) that is insensitive to economic fluctuation, especially considering that bank 1 does not default for any $\xi_T < \bar{\xi}_1$, as established in the previous section. Therefore, since $W_2 \leq \bar{W}_2$ implies that $\tilde{\xi}_2 < \tilde{\xi}_1$, it is optimal for bank 2 to enter financial distress for any realizations of the state price density in the interval $\tilde{\xi}_2 < \xi_T \leq \tilde{\xi}_1$. When $\tilde{\xi}_1 < \xi_T \leq \bar{\xi}_2$, bank 1 finds it optimal to resist default since bank 2 has no incentive to do so. This explains the extended region in which the equity of bank 2 sharply decreases below the default threshold (idiosyncratic default). Effectively, this sharp decrease, allows bank 2 to finance an increase in the default level once also bank 1 becomes insolvent (systemic default). The transfer of wealth from the idiosyncratic default states to the systemic default states makes the equity profile of bank 2 non-monotonic in the state price density. Indeed, at $\tilde{\xi}_1$, the value of the equity of bank 2 exhibits a upward jump equal to $(y^*_2 z_2 \tilde{\xi}_1)^{-1/\gamma_2} - (y^*_2 x_2 \tilde{\xi}_1)^{-1/\gamma_2}$. Moreover, in the systemic default region ($\xi_T > \bar{\xi}_1$), the curvature of the equity profile becomes flatter, highlighting bank 2’s desire to reduce its exposure to economic fluctuations. These unique features are brought about by the strategic interaction between the financial institutions. They would be absent in an economy without systemic externalities.

For completeness, in the following Corollary we show how to recover the equilibrium value of the assets and the debt at maturity from the equilibrium value of the equity.

**Corollary 2.** The optimal value of the assets and the optimal value of the debt of bank $i$ at time $T$ are given by

$$V^*_{iT} = \frac{1}{\beta_i} \left[ (W^*_{iT} + C^*_{iT}) + (1 - \beta_i)(W - W^*_{iT}) 1_{\{\xi_T \leq \tilde{\xi}_i\}} \right]$$  

$$D^*_{iT} = \frac{1 - \beta_i}{\beta_i} \left[ (W^*_{iT} + C^*_{iT}) + (W - W^*_{iT}) 1_{\{\xi_T \leq \tilde{\xi}_i\}} \right]$$  

respectively, where $C^*_{iT} \equiv C_{iT}(W^*_{iT})$.

**Proof.** See the Appendix.

We conclude this section by presenting some economically relevant examples of sources of heterogeneity among banks that lead to a unique equilibrium.
**HETEROGENEOUS BANKS: UNIQUE NASH EQUILIBRIUM**

solid line: $W^*_{1T}(\xi_T)$; dashed line: $W^*_{2T}(\xi_T)$; green line: pdf $(\xi_T|F_0)$

Panel A: $F_2 > F_1$

Panel B: $W_{20} < W_{10}$

Panel C: $\eta_2 > \eta_1 > 0$, high leverage

Panel D: $\beta_2 > \beta_1$

Panel E: $\gamma_2 > \gamma_1$, high leverage

Panel F: $\gamma_2 > \gamma_1$, low leverage

**Figure 2: Equilibrium horizon equity with heterogeneous banks**

**Parameter values.** Financial market (monthly): $r = 0.005$, $||\kappa|| = 0.2$. Horizon (years): $T = 5$. Banks: $W_{10} = W_{20} = 1$, $F_1 = F_2 = 7$, $\beta_1 = \beta_2 = 0.2$, $\gamma_1 = \gamma_2 = 2$, $\phi = 1.5\% W_{10}$, $\lambda = 5\%$, $\eta_1 = \eta_2 = 15\%$. Each panel contains a specific source of heterogeneity. Panel A: $F_2 = 9$. Panel B: $W_{20} = 0.778$, Panel C: $\eta_1 = 5\%$, $\eta_2 = 25\%$. Panel D: $\beta_2 = 0.25$. Panel E: $\gamma_2 = 4$. Panel F: $\gamma_2 = 4$, $F_1 = F_2 = 3.5$. 
Example 1: Heterogeneity in leverage. Consider bank 2 as more levered in the sense that \((F_2/W_{20}) - (F_1/W_{10}) > 0\). Bank 2 is the early-defaulter because its default boundary is higher. This case is illustrated in Panel A and Panel B, Figure 2.

Example 2: Heterogeneity in the objective function. Consider bank 2 as the one with a higher curvature in the objective function (more risk averse if \(u_i(\cdot)\) represents a utility function), \(\gamma_2 > \gamma_1\). Then, bank 2 is the early-defaulter if the level of leverage is high, whereas it becomes the late-defaulter if leverage is low. The intuition is the following: being more averse to the bad states of the world, bank 2 wants to transfer wealth from the good states to the bad states. If leverage is high, it means that default can occur in relatively good states; hence, by removing wealth from those states, bank 2 will increase the probability of default thus becoming the early-defaulter. If instead, leverage is low, default occurs only in very bad states of the world, exactly those states that bank 2 has “insured” by transferring wealth to. Therefore, it will decrease the probability of default, thus becoming the late-defaulter. These two case are illustrated in Panel E and Panel F, Figure 2.

Example 3: Heterogeneity in systemic costs. Consider bank 2 as to be more affected by systemic externalities, \(\eta_2 > \eta_1\). If leverage is high, systemic costs become relevant and induce bank 2 to transfer wealth from the good states to the bad states. When leverage is high, bank 2 increases the probability of default because it transfers wealth away from those states in which default gets triggered. Hence, bank 2 is the early-defaulter. This case is illustrated in Panel C, Figure 2.

Example 4: Heterogeneity in intangible assets/bargaining power (APR violations). Consider the case in which bank 2 has a higher fraction of intangible assets or its equityholders have a higher bargaining power, \(\beta_2 > \beta_1\). Since debtholders of bank 2 are going to seize a lower fraction of the assets in case of default, then the default boundary must increase, making bank 2 the early-defaulter. This case is illustrated in Panel D, Figure 2.

4 Results and Discussions

In the current section we analyze and discuss properties and implications of the unique strategic equilibrium derived in Section 3. For the sake of clarity, we consider heterogeneity in leverage in the form of different face values of debt \((F_j > F_i)\). \(^9\)

Moreover, we highlight the relevance of these results by comparing them with those derived from two benchmark models: (a) no cost of default; (b) no systemic cost of default. Both benchmarks

\(^9\)As Figure 2 shows, different sources of heterogeneity lead to very similar (unique) equilibrium profiles; therefore, equilibrium properties relevant to other source of heterogeneity are very similar to those presented in this section, and are available from the author upon request.
are special cases of our model where banks behave non-strategically. Benchmark (a) represents a frictionless economy where there are no costs associated to financial distress, $\phi = \lambda = \chi_i = \eta_i = 0$, for any $i$. In benchmark (b), instead, default is costly but systemic externalities are absent, $\chi_i = \eta_i = 0$, for any $i$. We refer to Appendix B for the solution of the two benchmark models.

In what follows, we adopt the upscripts $*$, $a$ and $b$ as notational convention for any equilibrium quantity related to the strategic model, benchmark (a) and benchmark (b), respectively.

### 4.1 Risk Taking Behavior: Optimal Asset Allocation

How are the banks’ risk exposures affected by the attempt to internalize (negative) systemic externalities? We provide an answer to this question by analyzing the optimal investment of the banks’ assets in risky securities. The next Proposition formally states the closed-form solution for the optimal asset allocation of the two financial institutions.

**Proposition 3.** The fractions of bank $i$’s assets invested in risky securities at time $t$ is given by

$$
\hat{\pi}_{it}^* = \hat{\pi}_{it}^* \cdot (\sigma')^{-1} \kappa \quad \text{where} \quad \hat{\pi}_{it}^* = -\frac{\xi_t}{V_{it}} \frac{\partial V_{it}}{\partial \xi_t}.
$$

W.l.o.g., let bank 1 be the late-defaulter and bank 2 the early-defaulter ($F_2 > F_1$). Then,

$$
\hat{\pi}_{1t}^* = \frac{1}{\gamma_1} + \frac{1}{V_{1t}} \left[ e^{-A_1(T-t)}(y_1^t\xi_t) - \frac{1}{\gamma_1} \left( \beta_1 n(-\bar{d}_1t(\xi_1)) - z_1^{1-1/\gamma_1} n(-\bar{d}_1t(\xi_1)) \right) \right] - \frac{e^{-r(T-t)}}{\beta_1 \gamma_1 \kappa \sqrt{T-t}} \left( \beta_1 W_1 \left[ 1 - \beta_1 N(-\bar{d}_1t(\xi_1)) \right] - z_1(W_1 - \phi) \left[ 1 - N(-\bar{d}_1t(\xi_1)) \right] \right),
$$

$$
\hat{\pi}_{2t}^* = \frac{1}{\gamma_2} + \frac{1}{V_{2t}} \left[ e^{-A_2(T-t)}(y_2^t\xi_t) - \frac{1}{\gamma_2} \left( \beta_2 n(-\bar{d}_2t(\xi_2)) + x_2^{1-1/\gamma_2} [n(-\bar{d}_2t(\xi_1)) - n(-\bar{d}_2t(\xi_2))] - z_2^{1-1/\gamma_2} n(-\bar{d}_2t(\xi_1)) \right) \right] - \frac{e^{-r(T-t)}}{\beta_2 \gamma_2 \kappa \sqrt{T-t}} \left( \beta_2 W_2 \left[ 1 - \beta_2 N(-\bar{d}_2t(\xi_2)) \right] - x_2(W_2 - \phi) \left[ N(-\bar{d}_2t(\xi_1)) - N(-\bar{d}_2t(\xi_2)) \right] - z_2(W_2 - \phi) \left[ 1 - N(-\bar{d}_2t(\xi_1)) \right] \right) - \frac{e^{-r(T-t)}}{\beta_2 \gamma_2 \kappa \sqrt{T-t}} \left( \beta_2 W_2 n(-\bar{d}_2t(\xi_1)) + x_2(W_2 - \phi) [n(-\bar{d}_2t(\xi_1)) - n(-\bar{d}_2t(\xi_2))] - z_2(W_2 - \phi) n(-\bar{d}_2t(\xi_1)) \right),
$$

where $N(\cdot)$ and $n(\cdot)$ are the cumulative distribution function and the probability density function of a standard-normal distribution, respectively. $V_{it}$, $A_i$, $\bar{d}_it(\cdot)$, $\bar{d}_it(\cdot)$ are reported in the Appendix.

**Proof.** See the Appendix. □
### Table 1: Unconditional Portfolio Moments

#### Parameter values.
Financial market (monthly): \( r = 0.005, ||\kappa|| = 0.2 \). Horizon (years): \( T = 5 \). Banks: \( W_{10} = W_{20} = 1 \), \( \beta_1 = \beta_2 = 0.2, \gamma_1 = \gamma_2 = 2, \phi = 1.5\% W_{10}, \lambda = 5\%, \eta_1 = \eta_2 = 15\% \). Source of heterogeneity: \( F_1 = 7, F_2 = 9 \).  

<table>
<thead>
<tr>
<th>Fraction of time: ( t/T )</th>
<th>0.05</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Correlation</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \rho_0(\hat{\pi}<em>{1t}^*, \hat{\pi}</em>{2t}^*) )</td>
<td>0.794</td>
<td>-0.044</td>
<td>-0.394</td>
<td>-0.463</td>
<td>-0.480</td>
</tr>
<tr>
<td>( \rho_0(\hat{\pi}<em>{1t}^a, \hat{\pi}</em>{2t}^a) )</td>
<td>0.999</td>
<td>0.990</td>
<td>0.970</td>
<td>0.930</td>
<td>0.845</td>
</tr>
<tr>
<td>( \rho_0(\hat{\pi}<em>{1t}^b, \hat{\pi}</em>{2t}^b) )</td>
<td>0.903</td>
<td>0.887</td>
<td>0.845</td>
<td>0.757</td>
<td>0.575</td>
</tr>
<tr>
<td><strong>Panel B: Variance</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{Var}<em>0(\hat{\pi}</em>{1t}^*)/\text{Var}<em>0(\hat{\pi}</em>{1t}^a) )</td>
<td>0.013</td>
<td>0.012</td>
<td>0.034</td>
<td>0.118</td>
<td>0.276</td>
</tr>
<tr>
<td>( \text{Var}<em>0(\hat{\pi}</em>{2t}^*)/\text{Var}<em>0(\hat{\pi}</em>{2t}^a) )</td>
<td>0.135</td>
<td>0.187</td>
<td>0.253</td>
<td>0.381</td>
<td>1.147</td>
</tr>
<tr>
<td>( \text{Var}<em>0(\hat{\pi}</em>{1t}^b)/\text{Var}<em>0(\hat{\pi}</em>{1t}^a) )</td>
<td>0.079</td>
<td>0.051</td>
<td>0.116</td>
<td>0.301</td>
<td>0.521</td>
</tr>
<tr>
<td>( \text{Var}<em>0(\hat{\pi}</em>{2t}^b)/\text{Var}<em>0(\hat{\pi}</em>{2t}^a) )</td>
<td>3.013</td>
<td>1.338</td>
<td>1.050</td>
<td>1.100</td>
<td>2.388</td>
</tr>
<tr>
<td><strong>Panel C: Mean</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{E}<em>0(\hat{\pi}</em>{1t}^*)/\text{E}<em>0(\hat{\pi}</em>{1t}^a) )</td>
<td>0.252</td>
<td>0.301</td>
<td>0.382</td>
<td>0.481</td>
<td>0.567</td>
</tr>
<tr>
<td>( \text{E}<em>0(\hat{\pi}</em>{2t}^*)/\text{E}<em>0(\hat{\pi}</em>{2t}^a) )</td>
<td>0.439</td>
<td>0.476</td>
<td>0.520</td>
<td>0.563</td>
<td>0.598</td>
</tr>
<tr>
<td>( \text{E}<em>0(\hat{\pi}</em>{1t}^b)/\text{E}<em>0(\hat{\pi}</em>{1t}^a) )</td>
<td>0.344</td>
<td>0.395</td>
<td>0.482</td>
<td>0.588</td>
<td>0.680</td>
</tr>
<tr>
<td>( \text{E}<em>0(\hat{\pi}</em>{2t}^b)/\text{E}<em>0(\hat{\pi}</em>{2t}^a) )</td>
<td>0.573</td>
<td>0.605</td>
<td>0.643</td>
<td>0.678</td>
<td>0.705</td>
</tr>
</tbody>
</table>

Proposition 3 immediately reveals that the optimal asset allocation of the two banks satisfies the two-fund separation theorem: both banks in equilibrium attain full diversification. Indeed, they invest in the same portfolio of risky assets, the mean-variance tangency portfolio (MVTP hereafter), with an exposure to it proportional to the elasticity of the equilibrium value of the assets with respect to economic fluctuations (represented by realizations of the state price density). This result highlights how the banks’ attempt to minimize systemic default costs does not distort the optimality of full diversification. This is, for instance, in contrast to the findings in Wagner (2011), where joint liquidation costs make (atomistic) agents prefer diversity at the expense of diversification. In our model full diversification is not compromised because both banks can move along the efficient frontier by altering (in a state-contingent manner) the fraction invested in the riskless bond, \( (1 - \hat{\pi}_{it}(\sigma'(\cdot)^{-1}\kappa)')1) \). By moving in different directions along the frontier, they attain diversity and preserve diversification.\(^{10}\)

\(^{10}\)We conjecture that in the absence of a riskless asset (hence incomplete markets) banks would choose two distinct risky portfolios. The analysis of such case is beyond the scope of this paper and it is left for future research.
The fact that both institutions invest in the (same) MVTP entails that at any point in time, conditional on the realization of a state of the world, the correlation between their portfolios is indeed perfect,

$$\rho_t(\pi^*_1t, \pi^*_2t) = \pm 1$$

(38)

for \(t \in (0, T)\). Unconditionally however, this correlation is less than one because the banks select optimal stochastic exposures that are not perfectly correlated. Equations (36) and (37) unveil the stochastic nature of the banks’ exposures as the source of heterogeneity in their trading strategies. Table 1 helps us to address the question of whether systemic costs of default reduce the (unconditional) correlation between these strategies. The answer is yes. Panel A presents the correlation coefficient as of time 0 between banks’ optimal time \(t\) risk exposure, for our strategic model and for the two benchmarks. A clear ranking appears:

$$\rho_0(\hat{\pi}^*_1t, \hat{\pi}^*_2t) < \rho_0(\hat{\pi}^b_1t, \hat{\pi}^b_2t) < \rho_0(\hat{\pi}^u_1t, \hat{\pi}^u_2t)$$

(39)

for \(t \in (0, T)\). The attempt to internalize systemic externalities, and hence to reduce systemic costs of joint default, induces the two banks to become more diverse. Table 1 also shows that the unconditional correlation decreases in all three models when maturity gets closer, becoming even negative in the strategic one. These results are qualitatively robust to different parameter values.

Figure 3 provides a graphical illustration of the investment behavior of the two financial institutions. In the top panels, exposures \(\hat{\pi}^*_it\) to the MVTP are plotted against realizations of the state price density for different points in time. In the bottom panels, instead, we plot the ratio between time \(t\) exposures coming from the strategic model and the two benchmarks (red line for benchmark (a), blue line for benchmark (b)). The following results emerge.

First, as shown in Table 1, the banks’ investment strategies become more uncorrelated as time to maturity decreases, and this is especially true for realizations of the state price density around the default thresholds (Panel A-C, Figure 3). The intuition is the following. Banks find it optimal to adopt diverse investment strategies if they reduce the likelihood of a systemic default. When \(t\) is close to \(T\) and economic conditions are very good (low values of \(\xi_t\)), it is very likely that at maturity such conditions will remain good; therefore, there is no need for diversity since any default is very unlikely. Analogously, when economic conditions are extremely bad (high values of \(\xi_t\)), it is very likely that at maturity bad conditions will persist: in this case, it would be too costly for both banks to avoid joint default, thus making diversity unnecessary. If instead, economic conditions are not so extreme (values of \(\xi_t\) around the default thresholds \(\bar{\xi}_2\) and \(\bar{\xi}_1\)), then diversity pays off. Trying to avoid joint default is not too costly and can be done by adopting negatively correlated investment strategies. Panel C in Figure 3 clearly highlights how diversity (correlation between investment strategies) changes with economic fluctuations, where the shaded areas correspond to extremely good (left area) and extremely bad (right area) economic conditions. Instead, when time
Figure 3: Banks’ optimal asset allocation

Parameter values. Financial market (monthly): $r = 0.005$, $||\kappa|| = 0.2$. Horizon (years): $T = 5$. Banks: $W_{10} = W_{20} = 1$, $\beta_1 = \beta_2 = 0.2$, $\gamma_1 = \gamma_2 = 2$, $\phi = 1.5\%W_{10}$, $\lambda = 5\%$, $\eta_1 = \eta_2 = 15\%$. Source of heterogeneity: $F_1 = 7$, $F_2 = 9$. 
to maturity is high (low $t$ as in Panel A), current economic conditions are not representative of the economic outlook at time $T$; therefore, it is optimal to wait before becoming diverse.

Second, conditional on becoming diverse (“medium” economic conditions and $t$ close to maturity), we observe interesting patterns in the risk taking decision. Consider bank 2, the early-defaulter. Panel C in Figure 3 shows that when both idiosyncratic default and systemic default are likely to occur (i.e., the intermediate unshaded area), bank 2’s risk taking at time $t$ exhibits two radically opposite behaviors. Indeed, it can be either very high, or negative. So, although defaulting is very likely, bank 2 could either invest a high fraction of the assets in risky securities, or, at the other extreme, take a short position in the MVTP.

To help explain the intuition behind this somehow surprising result, let us consider the equilibrium value of banks’ assets at time $T$ plotted in Figure 4, and recall that the optimal trading strategies are implemented to “deliver” those equilibrium asset values in each possible state of the world. Panel A of Figure 4 shows how the time $T$ value of the assets inherits all the properties of the equilibrium equity, described in details in Section 3. Because joint default is more costly (the negative externality), bank 2 wants to transfer wealth to the states that correspond to a systemic crisis ($\xi_T > \bar{\xi}_1$), in order to reduce the shortfall and hence the associated costs. This is attained by removing wealth from the most valuable states before $\bar{\xi}_1$. Therefore, this underlying mechanism produces two key properties of the model: (i) the sharp decrease in asset value in the idiosyncratic default region ($\tilde{\xi}_2 < \xi_T \leq \tilde{\xi}_1$), and (ii) the positive jump in correspondence to the threshold $\bar{\xi}_1$. Property (i) implies that the asset value is very sensitive and positively correlated to economic fluctuations, whereas property (ii) implies that it is negatively correlated.
Returning to the risk taking behavior at time $t$ (close to maturity), suppose $T$ is 5 years and we are 3 months away from maturity (as in Panel C, Figure 3). If economic prospects are not very good today, say $\xi_t$ just above $\tilde{\xi}_2$, then, most likely (since we are close to maturity) such prospects will not change much in 3 months and bank 2 will optimally enter (idiosyncratic) financial distress. The only way it can “deliver” an asset profile very correlated with the underlying uncertainty is by investing extensively in the market for risky securities today. To attain full diversification, a positive exposure to the MVTP is the optimal strategy. This explains the high risk taking. If, instead, economic conditions are worse today, say $\xi_t$ just below $\tilde{\xi}_1$, then the economy will face a systemic crisis with both banks defaulting if economic prospects slightly deteriorates. The only way bank 2 can “deliver” an asset profile that will jump upwards in such a case is by investing in financial securities that are negatively correlated with the economic fluctuations. To attain full diversification, a negative exposure to the MVTP is the optimal strategy. This explains the shorting behavior. The investment decisions of bank 1 can be explained in a similar fashion. However, high risk taking and short positions are absent since both idiosyncratic default and non-monotonic behavior (upward jump) are not part of the equilibrium value of the assets of bank 1. In summary, the banks adopt polarized and stochastic risk exposure. The early-defaulter is the radical bank. The late-defaulter is the conservative one.

Moreover, banks’ risk exposures when time to maturity is high are less volatile and some of the properties discussed above may not be present. For instance, the short position of bank 2 is absent in Panel B (Figure 3) because it would be too costly to short the MVTP two years and half before maturity. In other words, Panel B reveals that it is optimal to wait before taking extreme positions.

Third, compared to the non-strategic benchmarks, the unconditional mean of both banks’ risk exposure decreases, as reported in Panel C, Table 1. This is not surprising since the systemic externality considered in this paper imposes an extra burden to financial institutions. However, once we condition on time to maturity and economic conditions, we obtain the following. While bank 1’s risk taking is lower in the strategic model in (almost) all periods and states, bank 2’s behavior exhibits significantly more risk taking when time to maturity is low.¹¹ Not surprisingly, the need to finance the steep reduction in asset value in the idiosyncratic default region, absent in the two benchmark models, is the cause of the high exposure to the MVTP. Indeed, Panel F in Figure 3 shows how that the risk exposures in the strategic model can be around 2.5 and 3 times larger than in benchmark (a) and (b), respectively. Another feature that differentiates the strategic model from the benchmarks is the aforementioned shorting behavior. Since in both non-strategic models the optimal value of the assets at maturity is monotonic in the state price density (see Panel B and C in Figure 4), there is no need to take a (costly) short position in the MVTP. This
explains the difference in the variability of the asset allocation of bank 2, between the strategic model and the two benchmark models. The unconditional variance reported in Panel B, Table 1 confirms these findings.

4.2 Default Probabilities and Expected Shortfalls

In this section we study another set of important implications of the model. We analyze how internalizing systemic externalities affects the likelihood of idiosyncratic defaults and systemic crises. We then complement these findings with results on the expected shortfalls.

The equilibrium equity profiles of the two financial institutions, derived in Section 3, endogenously determine the banks’ optimal default thresholds in the state-space. For $\xi_T \leq \bar{\xi}_2$ neither of the banks default; for $\xi_2 < \xi_T \leq \bar{\xi}_1$ bank 2 (the early-defaulter) is insolvent; for $\xi_T > \bar{\xi}_1$ both banks default. The equilibrium default probabilities are defined only by the (absolute) positioning of the default thresholds in the state-space domain, as presented in the next Proposition.

**Proposition 4.** Let $D_i$ denote the financial distress event of bank $i$, and $\bar{D}_i$ its complement. W.l.o.g., let bank 1 be the late-defaulter and bank 2 the early-defaulter ($F_2 > F_1$). The strategic model produces the following time-$t$ probabilities:

(i) Marginal probability of default:

$$\mathbb{P}_t(D_1) = \mathcal{N}(d_t(\bar{\xi}_1)) \quad \text{and} \quad \mathbb{P}_t(D_2) = \mathcal{N}(d_t(\bar{\xi}_2)) $$  \hspace{1cm} (40)

(ii) Probability of systemic default:

$$\mathbb{P}_t(D_1 \cap D_2) = \mathcal{N}(d_t(\bar{\xi}_1)) $$  \hspace{1cm} (41)

(iii) Probability of idiosyncratic default:

$$\mathbb{P}_t(\cup_{j \neq i \in \{1,2\}} D_i \cap \bar{D}_j) = \mathcal{N}(d_t(\bar{\xi}_2)) - \mathcal{N}(d_t(\bar{\xi}_1)) $$  \hspace{1cm} (42)

(iv) Conditional probability of default:

$$\mathbb{P}_t(D_1|D_2) = \mathcal{N}(d_t(\bar{\xi}_1))/\mathcal{N}(d_t(\bar{\xi}_2)) \quad \text{and} \quad \mathbb{P}_t(D_2|D_1) = 1 $$  \hspace{1cm} (43)

where $\mathcal{N}(\cdot)$ represents the standard-normal cumulative distribution function. The function $d_t(\cdot)$ and the thresholds $\bar{\xi}_2$ and $\bar{\xi}_1$ are reported in the Appendix.

**Proof.** See the Appendix. \hfill \square
To appreciate the effect of the strategic interactions between the two financial institutions on the idiosyncratic and systemic default probabilities, we compare those of the strategic model with those of the two benchmarks. Similarly, we compare the expected shortfalls across the three models. A key question is: Are incentives to internalize systemic externalities effective in reducing the occurrence and the magnitude of systemic crises? The answer to this question is particularly relevant from a regulator’s viewpoint. Thanks to our flexible specification, the systemic costs of default can be readily interpreted as a macro-prudential regulation, allowing us to provide an answer.

Regarding default probabilities, we exploit the graphs in Figure 5 to highlight our findings. The following results emerge. Relative to the benchmarks, both banks are more likely to default, as illustrated in Panel A, where the black lines (the strategic model) are substantially higher than the corresponding colored lines (the benchmarks). This is true regardless of the time $t$ and state $\xi_t$ in which we evaluate these probabilities. An immediate implication of this result is that systemic default becomes more likely, since the event of a systemic crisis coincides with the default of bank 1 (Panel B).

Panel C reveals that, compared to the benchmarks, also the probability of an idiosyncratic default rises substantially. When close to maturity (Panel C3), however, this probability may become lower if the current economic conditions are particularly poor. If this is the case, an idiosyncratic default is less likely because the probability of systemic default is particularly high. Hence, the positioning of the black line to the left of benchmarks’ curves and its steeper profile confirms how the idiosyncratic default interval shifts to the left in the presence of negative systemic externalities.

Panel D considers the (non-degenerate) conditional probability of default defined in (43). Such a probability can be interpreted as the relative importance (in probabilistic terms) of a systemic crisis with respect to an idiosyncratic default:

$$P_t(D_1|D_2) = \frac{P_t(\text{systemic Default})}{P_t(\text{idiosyncratic Default}) + P_t(\text{systemic Default})}.$$  (44)

Panels D1-D2 show that, away from maturity, systemic externalities and strategic interaction lower the relative importance of a systemic crisis. However, close to maturity, the opposite holds.

Finally, Panel E considers the default correlation, defined as

$$\rho_t(D_1, D_2) \equiv \frac{P_t(D_1 \cap D_2) - P_t(D_1)P_t(D_2)}{\sqrt{P_t(D_1)[1 - P_t(D_1)]}\sqrt{P_t(D_2)[1 - P_t(D_2)]}}.$$  (45)

In line with the results on the optimal asset allocation, the default correlation in the strategic model lower for any time to maturity and current state of the economy.
**Figure 5: Banks’ default probabilities**

**Parameter values.** Financial market (monthly): $r = 0.005$, $||\kappa|| = 0.2$. Horizon (years): $T = 5$. Banks: $W_{10} = W_{20} = 1$, $\beta_1 = \beta_2 = 0.2$, $\gamma_1 = \gamma_2 = 2$, $\phi = 1.5\% W_{10}$, $\lambda = 5\%$, $\eta_1 = \eta_2 = 15\%$. Source of heterogeneity: $F_1 = 7$, $F_2 = 9$. 
This set of findings does not easily reconcile with the ex-ante objective of macro-prudential regulation to reduce the likelihood of a systemic event. We provide next a simple explanation for these results. We argue that, compared to the benchmarks, default thresholds and hence default probabilities, are affected by the net result of a substitution and an income effect. The substitution effect captures the banks’ desire to transfer wealth from the good states to the bad states in order to reduce the potential exposure to high cost of systemic default. Therefore, the substitution effect translates into a lower probability of joint default, since banks attempt to internalize systemic externalities. However, although it incentivizes a transfer of wealth across states, a high cost of systemic default also represents a burden to the banks’ budget constraint. Equivalent to an overall drop in the banks’ capital, the income effect causes a higher probability of joint default. The equilibrium outcome is thus the net of these two opposing effects. The two competing effects are “extracted” from the ratio of systemic default thresholds across models:

$$\frac{\bar{\xi}_1}{\xi_1} = \frac{\left(\frac{\alpha_1}{W_1^{-\gamma_1} z_1}\right)}{\left(\frac{y^1_1}{y^a_1}\right)}$$  \hspace{1cm} (46)

$$\frac{\bar{\xi}_1}{\xi_1} = \frac{\left(\frac{x_1}{z_1}\right)}{\left(\frac{y^b_1}{y^1_1}\right)}$$  \hspace{1cm} (47)

Hence, we define the substitution and income effects as follows,

$$\varsigma^a \equiv \alpha_1/(W_1^{-\gamma_1} z_1), \quad \iota^a \equiv y^a_1/y^1_1, \quad \varsigma^b \equiv x_1/z_1, \quad \iota^b \equiv y^b_1/y^1_1,$$  \hspace{1cm} (48)

where $\alpha_1$ solves equation (24). The substitution effect, with respect to benchmark $k$, dominates if $\varsigma^k > \iota^k$ for $k \in \{a, b\}$. The income effect dominates otherwise.

Our model predicts that for financial institutions with moderate and high leverage ratios, the income effect always dominates, regardless of the magnitude of the default externalities. These dynamics lead to the documented increase in probabilities of idiosyncratic and systemic default. We assess this effect to be economically significant, considering the prevalence of extremely high leverage ratios among financial institutions. Figure 6 illustrates this result. Each bar-chart plots the pair of substitution (bright bar) and income (dark bar) effect for different leverage ratios. In Panel B and Panel C default costs are lower and higher than in Panel A, respectively. Compared to both benchmarks (top panels refers to benchmark (a), bottom panels to benchmark (b)), we confirm that: (i) the income effect dominates the substitution effect for medium-high leverage ratios; (ii) this relationship is not affected by the magnitude of default costs.

Default probabilities describe the likelihood of a default, but they are not informative on the extent of default. For this reason, we complete our analysis by examining expected shortfalls. Specifically, we consider those arising from an idiosyncratic and a systemic default.
Figure 6: Substitution and income effects

Parameter values. Financial market (monthly): \( r = 0.005, ||\kappa|| = 0.2 \). Horizon (years): \( T = 5 \). Banks: \( W_{10} = W_{20} = 1, \beta_1 = \beta_2 = 0.2, \eta_1 = \eta_2 = 2, \phi = 1.5\% W_{10}, \lambda = 5\%, \eta_1 = \eta_2 = 15\% \). Source of heterogeneity: \( F_2 = 9/7 \times F_1 \).
Proposition 5. W.l.o.g., let bank 1 be the late-defaulter and bank 2 the early-defaulter \((F_2 > F_1)\). The idiosyncratic and systemic expected shortfalls at time \(t\) are given by

\[
\begin{align*}
\text{IES}_t &= \mathbb{E}_t \left[ \frac{\xi_t}{\xi_t} (F_2 - D_{2T}) \mathbb{1}_{\{\xi_t < \xi_t \leq \xi_t\}} \right] \\
&= \left( \frac{1 - \beta_2}{\beta_2} \right) \left[ x_2(W_2 - \phi) e^{-r(T-t)} \left\{ \mathcal{N}(-\tilde{d}_t(\xi)) - \mathcal{N}(\tilde{d}_t(\xi_2)) \right\} \\
&\quad - x_2(y_2 x_2 \xi_t) - \frac{1}{2} e^{-A_2(T-t)} \left\{ \mathcal{N}(-\tilde{d}_2(\xi_1)) - \mathcal{N}(\tilde{d}_2(\xi_2)) \right\} \right], \\
\end{align*}
\]

\(49\)

\[
\begin{align*}
\text{SES}_t &= \mathbb{E}_t \left[ \frac{\xi_t}{\xi_t} \left\{ (F_1 - D_{1T}) + (F_2 - D_{2T}) \right\} \mathbb{1}_{\{\xi_t > \xi_t\}} \right] \\
&= \sum_{i=1,2} \left( F_i e^{-r(T-t)} - D_{it} \right) - \text{IES}_t,
\end{align*}
\]

\(50\)

respectively, where

\[
\begin{align*}
D_{1t} &= \left( \frac{1 - \beta_1}{\beta_1} \right) \left[ e^{-A_1(T-t)} (y_1^* \xi_t) - \frac{1}{\gamma_1} \left( \frac{1}{\gamma_1} - \mathcal{N}(\tilde{d}_1(\xi)) \right) \right] \\
&\quad + e^{-r(T-t)} \left( W_1 - z_1(W_1 - \phi) \left[ 1 - \mathcal{N}(\tilde{d}_1(\xi)) \right] \right), \\
\end{align*}
\]

\(51\)

\[
\begin{align*}
D_{2t} &= \left( \frac{1 - \beta_2}{\beta_2} \right) \left[ e^{-A_2(T-t)} (y_2^* \xi_t) - \frac{1}{\gamma_2} \left( \frac{1}{\gamma_2} - \mathcal{N}(\tilde{d}_2(\xi)) \right) \right] \\
&\quad + e^{-r(T-t)} \left( W_2 - x_2(W_2 - \phi) \left[ \mathcal{N}(\tilde{d}_2(\xi)) \right] \right), \\
\end{align*}
\]

\(52\)

\(\mathcal{N}(\cdot)\) represents the standard-normal cumulative distribution function. \(A_i, \tilde{d}_i(\cdot), \tilde{d}_t(\cdot)\) are reported in the Appendix.

Proof. See the Appendix.

By means of Figure 7, we highlight the following results. Idiosyncratic expected shortfall are significantly higher in the strategic model than in the benchmarks. This precisely reflects the transfer of wealth that the early-defaulter engages in. However, when time to maturity is low (Panel C), the expected shortfall may become lower if the current economic conditions are particularly poor. This is justified by the fact that, as reported in Figure 5 (Panel C3), the probability of an idiosyncratic default becomes very low. While we have shown that the probabilities of systemic default are unambiguously higher for the strategic model, the same does not apply to the systemic expected shortfall. Indeed, as shown in Panel D-F, by internalizing the externalities of default, the
expected shortfall is always below the expected shortfall of the benchmarks in the most adverse states. Thus, we can conclude that also the expected losses given default must be lower under extremely adverse economic conditions, considering that systemic default probabilities are higher overall. We still find though that around the default thresholds, the expected systemic shortfall may become higher then in the benchmark cases.

We believe the results presented in this section are indicative of a friction between systemic default probabilities on one hand and systemic losses on the other hand, important to a regulator concerned with the design of a macro-prudential framework.
4.3 Debt Pricing: Credit Spreads and CDS

Since Merton (1974), structural models of credit risk have been developed with the scope of deriving the value of debt issued by a firm, by means of a contingent claim (no-arbitrage) analysis. In this section, we discuss the strategic equilibrium implications on the value of credit spreads and credit default swap premiums for the two banks and relate them to the results presented in the previous sections. Proposition 6 presents the closed-form equilibrium values.

Proposition 6. W.l.o.g., let bank 1 be the late-defaulter and bank 2 the early-defaulter \((F_2 > F_1)\). The credit spreads and the credit default swap premium written on the bond of bank \(i\) are given by

\[
CS_{it} = \frac{1}{T-t} \ln \left( \frac{F_i}{D_{it}} \right) - r \tag{53}
\]

\[
CDS_{it} = \frac{E_t[\xi_T / \xi_t] (F_i - D_{iT}) 1_{\{D_{it}\}}}{E_t[\xi_T / \xi_t] 1_{\{\bar{D}_{it}\}}} \tag{54}
\]

respectively, where \(D_{it}\) is provided in equations (51) and (52). Hence,

\[
\frac{CDS_{1t}}{F_1} = 1 - e^{-CS_{1t}(T-t)} \frac{1}{N(-d_t(\bar{\xi}_1))}, \quad \frac{CDS_{2t}}{F_2} = 1 - e^{-CS_{2t}(T-t)} \frac{1}{N(-d_t(\bar{\xi}_2))}. \tag{55}
\]

Proof. See the Appendix.

Debt is a claim on the value of the assets, which in the equilibrium of our model are determined endogenously by the strategic risk taking of the banks. Therefore the value of the debt is directly linked to the primitives of the strategic interactions, as is clear from (51) and (52). This facilitates the interpretation of the credit spread dynamics.

For both the early- and the late-defaulter, we find that the value of the credit spreads is lower than in the benchmark models for adverse states of the world, at any time to maturity (Panel A-C, Figure 8). This is driven by the wealth transfers that the banks implement, from good states to systemic states, in order to minimize the burden of joint default. These wealth transfers have a trade-off: by removing wealth from the good states, the banks’ bonds become riskier in spite of favorable economic conditions. The patterns in Panel A-C of Figure 8 highlight the close connection between the bonds’ prices and the banks’ strategic risk taking. In particular, we want to emphasize the risk-profile of bank 2’s credit spreads close to maturity (Panel C, Figure 8). Indeed, these credit spreads inherit the distinctive features of bank 2’s risk exposure (Panel C, Figure 3). In the vicinity of threshold \(\bar{\xi}_2\), it is very likely that the idiosyncratic default of bank 2 occurs, implying a large shortfall on the debt repayment. This is naturally reflected in the credit spreads. In the vicinity of threshold \(\bar{\xi}_1\) the credit spreads behave non-monotonically: lower credit spreads are associated with
Figure 8: Banks’ debt: credit spreads and cds

Parameter values. Financial market (monthly): \( r = 0.005, ||\kappa|| = 0.2 \). Horizon (years): \( T = 5 \). Banks: \( W_{10} = W_{20} = 1, \beta_1 = \beta_2 = 0.2, \gamma_1 = \gamma_2 = 2, \phi = 1.5\%W_{i0}, \lambda = 5\%, \eta_1 = \eta_2 = 15\% \). Source of heterogeneity: \( F_1 = 7, F_2 = 9 \).
worse states. The shorting position, that guarantees the financing of a higher wealth level in case of a systemic default, effectively makes the banks’ debt less risky.

The credit default swap spreads in our model reflect the price one needs to pay for protection against the possibility of time $T$ default of the reference bank. This price is positively related to both the probability of default and the credit spread, as revealed in (55). We document in the lower panels of Figure 8 that the price to pay for protection against default is higher in the strategic model than in the benchmarks. This is also true for the systemic states where the credit spreads of the banks are markedly lower than for the benchmark models. This indicates that the effect of the increased default probabilities dominates on the equilibrium CDS prices.

5 Concluding Remarks

Arguably, the financial sector is dominated by a small set of highly levered financial institutions with strong interlinkages, giving rise to strategic interactions. This paper analyzes the strategic risk taking of two such financial institutions, when systemic default induces externalities that amplify the cost of financial distress. We develop a structural model of credit risk in which, for a given capital structure, the asset value dynamics are endogenously determined by the optimal portfolio allocation.

We derive a unique strategic equilibrium in which heterogeneous banks adopt polarized and stochastic risk exposure, without sacrificing full diversification. In the presence of systemic externalities, both banks care about financing a sufficiently high level of wealth in adverse states. To this purpose, the conservative bank reduces its risk exposure, whereas the radical bank optimally gambles on positive and negative outcomes, by taking either a large long or a short position in risky securities. The underlying economic mechanism increases the likelihood of systemic default. We believe that our analysis of systemic crises, both in terms of likelihood and expected shortfall, within a workhorse dynamic asset allocation framework is new and delivers a rich set of implications regarding financial institutions strategic risk taking.

The tractable framework developed in this paper provides a platform to investigate other relevant questions. For instance, evaluating the effectiveness of different proposals to regulate systemic risk in a context of externalities represents one possible extension. Moreover, in this paper we show that a multiplicity of equilibria arises when financial institutions are very homogeneous. Thus, selecting amongst these equilibria and understanding their interplay with macro-prudential policies represents another promising direction for future research. Finally, one could bring the analysis to a general equilibrium level in order to study the impact of strategic interaction and systemic externalities on asset prices.
Appendix

A Proofs

Proof of Lemma 1. For each default region, substitute in (8) the relevant value of the assets as a function of \((W_{it}, W_{jt})\). Straightforward algebra leads to (16).

\[ \text{Proof of Proposition 1.} \text{ We solve for bank } i's \text{ best response function considering a generic objective function } u_i(\cdot) \text{ that satisfies the properties in Remark 1. The problem faced by bank } i \text{ can be restated using Lagrangian as} \]

\[
\max_{W_{it}, y_i} \mathbb{E}\left[u_i(W_{it}) + y_i[W_{i0} - \xi_T(W_{it} + C_{it})]\right] \tag{A.1}
\]

For each state \(\xi_T\):

\[
\hat{W}_{it} = \arg \max u_i(W_{it}) + y_iW_{i0} - y_i\xi_T W_{it} - y_i\xi_T C_{it}
\]

\[
= \arg \max u_i(W_{it}) - y_i\xi_T W_{it} - y_i\xi_T \left\{[x_i\phi + (1 - x_i)(W_{i} - W_{it})] \mathbb{I}_{\{W_{it} < W_i \land W_{jt} \geq W_j\}}
\]

\[
+ [z_i\phi + (1 - z_i)(W_{i} - W_{it})] \mathbb{I}_{\{W_{it} < W_i \land W_{jt} < W_j\}} \right\} \tag{A.2}
\]

Let \(h_i(W_{it})\) denote the objective function on the RHS of (A.2). \(h_i(W_{it})\) is not concave in \(W_{it}\) because of the discontinuity in \(W_i\), and it exhibits possible maxima at:

\[
I_i(y_i\xi_T) \quad \text{if} \quad I_i(y_i\xi_T) \geq W_i \quad \tag{A.3}
\]

\[
W_i \quad \tag{A.4}
\]

\[
I_i(y_i x_i\xi_T) \quad \text{if} \quad I_i(y_i x_i\xi_T) < W_i \land W_{jt} \geq W_j \quad \tag{A.5}
\]

\[
I_i(y_i z_i\xi_T) \quad \text{if} \quad I_i(y_i z_i\xi_T) < W_i \land W_{jt} < W_j \quad \tag{A.6}
\]

where \(I_i(\cdot)\) is the inverse function of \(u_i'(\cdot)\). In what follows, we show for which regions of the domain of \(\xi_T\) the above candidates are indeed a global maximum. Note that, since \(I_i(\alpha)\) is decreasing in \(\alpha\) and \((x_i, z_i) \in (0, 1)\):

- when \(I_i(y_i\xi_T) \geq W_i\), then the maxima at \(I_i(y_i x_i\xi_T)\) and \(I_i(y_i z_i\xi_T)\) are not feasible since \(I_i(y_i z_i\xi_T) > I_i(y_i x_i\xi_T) > W_i\); hence the possible candidates are \(I_i(y_i\xi_T)\) and \(W_i\);
- when \(I_i(y_i x_i\xi_T) < W_i\), then the maximum at \(I_i(y_i\xi_T)\) is not feasible since \(I_i(y_i\xi_T) < W_i\); hence the possible candidates are \(I_i(y_i x_i\xi_T)\) and \(W_i\);
- when \(I_i(y_i z_i\xi_T) < W_i\), then the maximum at \(I_i(y_i\xi_T)\) is not feasible since \(I_i(y_i\xi_T) < W_i\); hence the possible candidates are \(I_i(y_i z_i\xi_T)\) and \(W_i\).

Therefore, we can distinguish among the following cases:

\[39\]
(i) \( h_i(I_i(y_i, \xi_T)) \) is a global maximum if the following two conditions are satisfied:

\[
I_i(y_i, \xi_T) - W_i > 0 \tag{A.7}
\]

\[
h_i(I_i(y_i, \xi_T)) - h_i(W_i) \geq 0 \tag{A.8}
\]

Since for any \( W_{iT} \geq W_i \) bank \( i \) does not default, (A.7) implies (A.8). Hence, \( \bar{W}_{iT} = I_i(y_i, \xi_T) \) if \( \xi_T \leq \bar{\xi}_i \) where \( \bar{\xi}_i \equiv u_i'(W_i)/y_i \).

(ii) When \( W_{jT} \geq W_j \), \( h_i(I_i(y_i, x_i, \xi_T)) \) is a global maximum if the following two conditions are satisfied:

\[
I_i(y_i, x_i, \xi_T) - W_i < 0 \tag{A.9}
\]

\[
h_i(I_i(y_i, x_i, \xi_T)) - h_i(W_i) \geq 0 \tag{A.10}
\]

Equation (A.9) implies that \( \xi_T > \bar{\xi}_i/x_i \), and (A.10) can be written as \( f_i(y_i, x_i, \xi_T) \leq 0 \) where

\[
f_i(\alpha) \equiv \left[ u_i(W_i) - u_i(I_i(\alpha)) \right] \frac{1}{\alpha} + I_i(\alpha) - W_i + \phi \tag{A.11}
\]

\[
f_i(y_i, \bar{\xi}_i) \equiv \phi \tag{A.12}
\]

\[
\frac{\partial f_i(\alpha)}{\partial \alpha} < 0 \quad \text{if} \quad I_i(\alpha) < W_i \tag{A.13}
\]

Let \( \bar{\xi}_i \) be such that \( f_i(y_i, \bar{\xi}_i, \bar{\xi}_i) = 0 \) and \( \bar{\xi}_i > \bar{\xi}_i/x_i \). From (A.12) and (A.13) we can conclude that \( \bar{\xi}_i \) exists and it is unique. Hence, \( \bar{W}_{iT} = I_i(y_i, x_i, \xi_T) \) if \( \xi_T > \bar{\xi}_i \). Since \( f_i(u_i'(W_i - \phi)) > 0 \), in case of default, \( \bar{W}_{iT} = I_i(y_i, x_i, \xi_T) < W_i - \phi \) for any \( \xi_T > \bar{\xi}_i \). This proves that the optimal equity of bank \( i \) does not take values in the interval \( [W_i - \phi, W_i] \).

(iii) Symmetrically, when \( W_{jT} < W_j \), \( h_i(I_i(y_i, z_i, \xi_T)) \) is a global maximum if the following two conditions are satisfied:

\[
I_i(y_i, z_i, \xi_T) - W_i < 0 \tag{A.14}
\]

\[
h_i(I_i(y_i, z_i, \xi_T)) - h_i(W_i) \geq 0 \tag{A.15}
\]

Equation (A.14) implies that \( \xi_T > \bar{\xi}_i/z_i \), and (A.15) can be written as \( f_i(y_i, z_i, \xi_T) \leq 0 \) where \( f_i(\cdot) \) is defined above in (A.11). Let \( \bar{\xi}_i \) be such that \( f_i(y_i, z_i, \bar{\xi}_i) = 0 \) and \( \bar{\xi}_i > \bar{\xi}_i/z_i \). From (A.12) and (A.13) we can conclude that \( \bar{\xi}_i \) exists and it is unique. Hence, \( \bar{W}_{iT} = I_i(y_i, z_i, \xi_T) \) if \( \xi_T > \bar{\xi}_i \).

(iv) When \( W_{jT} \geq W_j \), \( h_i(W_i) \) is a global maximum if either

\[
I_i(y_i, \xi_T) - W_i < 0 \tag{A.16}
\]

\[
I_i(y_i, x_i, \xi_T) - W_i \geq 0 \tag{A.17}
\]

that is if \( \xi_i < \xi_T \leq \bar{\xi}_i/x_i \), or

\[
I_i(y_i, x_i, \xi_T) - W_i < 0 \tag{A.18}
\]

\[
h_i(I_i(y_i, x_i, \xi_T)) - h_i(W_i) > 0 \tag{A.19}
\]

that is if \( \bar{\xi}_i/x_i < \xi_T \leq \bar{\xi}_i \). Hence, \( \bar{W}_{iT} = W_i \) if \( \bar{\xi}_i < \xi_T \leq \bar{\xi}_i \).
(v) When $W_j T < W_j$, $h_i(W_i)$ is a global maximum if either
\[ I_i(y_i \xi T) - W_i < 0 \quad \text{(A.20)} \]
\[ I_i(y_i z_i \xi T) - W_i \geq 0 \quad \text{(A.21)} \]
that is if $\xi_i < \xi T \leq \xi_i/z_i$, or
\[ I_i(y_i z_i \xi T) - W_i < 0 \quad \text{(A.22)} \]
\[ h_i(I_i(y_i z_i \xi T)) - h_i(W_i) > 0 \quad \text{(A.23)} \]
that is if $\xi_i/z_i < \xi T \leq \bar{\xi}_i$. Hence, $W_i$ is a global maximizer if $\xi_i < \xi T \leq \bar{\xi}_i$. 

Putting together all the five cases and considering an isoelastic objective function as in (5), where
\[ I_i(x) = x^{-\frac{1}{\gamma_i}}, \quad \text{(A.24)} \]
we obtain (22). We have already shown that $\xi_i < \bar{\xi}_i$, and it is straightforward to see that $\bar{\xi}_i < \bar{\xi}_i$, since $x_i > z_i$.

Given the optimal solution $\bar{W}_iT(W_jT; y_i)$, $y_i$ is set such that the static budget constraint in (19) holds with equality. Let $\bar{W}_iT$ be any feasible solution (i.e., that satisfies the static budget constraint), then we can show that
\[ E[u_i(\bar{W}_iT)] - E[u_i(\bar{W}_iT)] = E[u_i(\bar{W}_iT) - y_i W_i0] - E[u_i(\bar{W}_iT) - y_i W_i0] \bigg) \]
\[ \geq E[u_i(\bar{W}_iT) - y_i \xi T(\bar{W}_iT + C_iT(\bar{W}_iT))] - E[u_i(\bar{W}_iT) - y_i \xi T(\bar{W}_iT + C_iT(\bar{W}_iT))] \]
\[ \geq 0 \quad \text{(A.25)} \]

The first inequality follows from (19) holding with equality for $\bar{W}_iT$. The second inequality follows from (A.2).

\begin{lemma}
If bank $i$ defaults, then
\[ W_iT + C_iT < W_i \quad \text{(A.26)} \]
\end{lemma}

\begin{proof}
Let us consider first the case $W_j T \geq W_j$:
\[ W_iT + C_iT = I_i(y_i x_i \xi T) + [x_i \phi + (1 - x_i)(W_i - I_i(y_i x_i \xi T))] \quad \text{(A.27)} \]
\[ = W_i + x_i [\phi - W_i + I_i(y_i x_i \xi T)] \quad \text{(A.28)} \]
Since $I_i(\alpha)$ is decreasing in $\alpha$, by using the definition of $\bar{\xi}_i$ in (23), we can conclude that
\[ \phi - W_i + I_i(y_i x_i \xi T) < \phi - W_i + I_i(y_i x_i \xi_i) = -\frac{[u_i(W_i) - u_i(I_i(y_i x_i \xi_i))] - 1}{y_i x_i \xi_i} < 0 \quad \text{(A.29)} \]
Hence, $W_iT + C_iT < W_i$. A symmetric proof, which makes use of the definition of $\bar{\xi}_i$ in (23), holds for the case $W_j T < W_j$. 
\end{proof}
Proof of Proposition 2. As for the proof of Proposition 1, we consider a generic objective function $u_i(\cdot)$. Consider any realization of $\xi_T$ in the set $[\xi_i, \tilde{\xi}_i] \cap (\tilde{\xi}_j, \bar{\xi}_j]$, where the thresholds $\xi_j, \tilde{\xi}_i$ are defined in (23). From the best response functions in (22), it follows that the Nash equilibrium of the game for a fixed level of $\xi_T$ (which we refer as the $\xi_T$-game hereafter) is multiple. Hence, a unique pure-strategy equilibrium of the strategic game requires that the set $[\xi_i, \tilde{\xi}_i] \cap (\tilde{\xi}_j, \bar{\xi}_j]$ is empty: $\tilde{\xi}_j < \xi_i$. In turn, this requires that

$$y_j > \bar{y}_j \quad \text{where} \quad \bar{y}_j \equiv y_i \left( \frac{\alpha_j x_i}{\alpha_i z_j} \right). \quad (A.30)$$

Let assume that $y_j > \bar{y}_j$, then the Nash equilibrium of each $\xi_T$-game, for any $\xi_T$, is unique. This can be written as

$$W^*_{iT}(y_i) = \begin{cases} 
I_i(y_i, \xi_T) & \text{if} \quad \xi_T \leq \xi_i \\
W_i & \text{if} \quad \xi_i < \xi_T \leq \tilde{\xi}_i \\
I_i(y_i z_i, \xi_T) & \text{if} \quad \tilde{\xi}_i < \xi_T \
\end{cases}$$

$$W^*_{jT}(y_j, y_i) = \begin{cases} 
I_j(y_j, \xi_T) & \text{if} \quad \xi_T \leq \xi_j \\
W_j & \text{if} \quad \xi_j < \xi_T \leq \tilde{\xi}_j \\
I_j(y_j, y_i, \xi_T) & \text{if} \quad \tilde{\xi}_j < \xi_T \leq \bar{\xi}_j \\
I_j(y_j z_j, \xi_T) & \text{if} \quad \bar{\xi}_j < \xi_T, \quad (A.31) 
\end{cases}$$

This corresponds to a unique pure-strategy equilibrium of the strategic game if and only if the Lagrange multipliers $(y_i^*, y_j^*)$ that make the budget constraints binding

$$E \left[ \xi_T (W^*_{iT}(y_i^*) + C_{iT}(W^*_{jT}(y_j^*))) \right] = W_{i0} \quad (A.32)$$

$$E \left[ \xi_T (W^*_{jT}(y_j^*, y_i^*) + C_{jT}(W^*_{jT}(y_j^*, y_i^*))) \right] = W_{j0}, \quad (A.33)$$

are such that the assumed condition is indeed satisfied: $y_j^* > y_i^* \left( \frac{\alpha_j x_i}{\alpha_i z_j} \right)$. To verify this we proceed as follows. First notice that the budget constraint in (A.32) depends only on $y_i$ and not on $y_j$. This allows us to get $y_i^*$ independently of $y_j$. Then, by means of the following budget-constraint operator

$$BC_j(y_j) \equiv E \left[ \xi_T (W^*_{jT}(y_j, y_i^*) + C_{jT}(W^*_{jT}(y_j, y_i^*))) \right] \quad (A.34)$$

we define

$$W_{j0} \equiv BC_j \left( y_i \left( \frac{\alpha_j x_i}{\alpha_i z_j} \right) \right) \quad (A.35)$$

By monotonicity of the operator in (A.34), for which we omit the proof, we can conclude that

$$W_{j0} < W_{j0} \iff y_j^* > y_i^* \left( \frac{\alpha_j x_i}{\alpha_i z_j} \right) \quad (A.36)$$

When this condition holds for only one of the two banks, the equilibrium is unique.

Proof of Corollary 1. Part (i) follows from Proposition 2. Define $\bar{y}_j \equiv y_i(x/z) > y_j$. It follows from (A.31) that $W^*_{iT}(y_j) > W^*_{jT}(\bar{y}_j, y_i)$ for any $\xi_T$. Hence it must be that $W_{0} > W_{j0}$. This clearly holds for both banks. To prove part (ii), suppose by contradiction that a symmetric equilibrium exists: $(W^*_{1T}, y_1^*) = (W^*_{2T}, y_2^*)$. Then, since $y_1^* = y_2^*$,

$$\bar{\xi} \equiv \bar{\xi}_1 = \bar{\xi}_2, \quad \tilde{\xi} \equiv \tilde{\xi}_1 = \tilde{\xi}_2, \quad \bar{\xi} \equiv \bar{\xi}_1 = \bar{\xi}_2 \quad \text{(A.37)}$$
To conclude that this can not be part of an equilibrium of the strategic game played by the two banks, it is enough to show that there exists a realization of the state of nature (the SPD) for which the Nash equilibrium is not symmetric. So, consider any realization $\xi_T$ in the interval $\xi < \xi_T \leq \bar{\xi}$; according to Equation (22) in Proposition 1, if $W_{jT} \geq W$, the optimal response of bank $i$ is to default, $W_{iT} < W$, whereas if $W_{jT} < W$, the optimal response of bank $i$ is not to default, $W_{iT} \geq W$. This contradicts the initial assumption that $W_{1T} = W_{2T}$ can be part of an equilibrium.

Proof of Corollary 2. The optimal value of the assets follows from (11), (12), (14), and Lemma 1. The optimal value of the debt follows from: $D_{iT}^* = V_{iT}^* - W_{iT}^* - C_{iT}^*$.

Lemma 3. The state price density follows a Geometric Brownian Motion:

$$\frac{d\xi_t}{\xi_t} = -rdt - \kappa dw_t$$

Then, the following results hold:

$$E_t[1_{\{\xi_T \leq H\}}] = N(-d_t(H))$$
$$E_t[\xi_T 1_{\{\xi_T \leq H\}}] = \xi_t e^{-r(T-t)} N(-\bar{d}_t(H))$$
$$E_t[\xi_T^{(\gamma_i-1)/\gamma_i} 1_{\{\xi_T \leq H\}}] = \xi_t^{(\gamma_i-1)/\gamma_i} e^{-A_t(T-t)} N(-\hat{d}_t(H))$$

where $N(\cdot)$ is the standard-normal cumulative distribution function and

$$d_t(H) = \frac{\ln (\xi_t / H) - (r - (||\kappa||^2/2))(T-t)}{||\kappa||\sqrt{T-t}}$$
$$\bar{d}_t(H) = d_t(H) + ||\kappa||\sqrt{T-t}$$
$$\hat{d}_t(H) = \bar{d}_t(H) - \frac{||\kappa||}{\gamma_i} \sqrt{T-t}$$
$$A_t = \left( \frac{\gamma_i - 1}{\gamma_i} \right) \left[ r + \frac{||\kappa||^2}{2\gamma_i} \right]$$

Proof. The results follow from the conditional expectation of a log-normal random variable.

Proof of Proposition 3. To derive the optimal asset allocation we first compute the time $t$ value of the equity and of the assets:

$$W_{iT}^* = E_t \left[ \frac{\xi_T}{\xi_t} (W_{iT}^* + C_{iT}^*) \right]$$
$$V_{iT}^* = \frac{1}{\beta_i} E_t \left[ \frac{\xi_T}{\xi_t} (W_{iT}^* + C_{iT}^*) + (1 - \beta_i) \frac{\xi_T}{\xi_t} (W - W_{iT}^*) 1_{\{\xi_T \leq \xi_t\}} \right].$$
Specifically,

\[ W_{it}^* = e^{-r(T-t)} \left\{ W_i \left[ 1 - \mathcal{N}(-\tilde{d}_i(x_{it})) \right] - z_1(W_{it} - \phi) \left[ 1 - \mathcal{N}(-\tilde{d}_i(x_{it})) \right] \right\} \\
+ e^{-A_i(T-t)}(y_{it}^* x_{it}) \frac{1}{\beta_i} \left\{ \mathcal{N}(-\tilde{d}_it(x_{it})) + z_1 \beta_i^{-1} \left[ 1 - \mathcal{N}(\tilde{d}_it(x_{it})) \right] \right\}, \]  
(A.47)

\[ W_{2t}^* = e^{-r(T-t)} \left\{ W_2[1-\mathcal{N}(-\tilde{d}_2(x_{2t}))]-z_2(W_2 - \phi)\left[ 1 - \mathcal{N}(-\tilde{d}_2(x_{2t})) \right] - z_2(W_2 - \phi) \left[ 1 - \mathcal{N}(-\tilde{d}_2(x_{2t})) \right] \right\} \\
+ e^{-A_2(T-t)}(y_{2t}^* x_{2t}) \frac{1}{\beta_2} \left\{ \mathcal{N}(-\tilde{d}_{2t}(x_{2t})) + z_2 \beta_2^{-1} \left[ 1 - \mathcal{N}(\tilde{d}_{2t}(x_{2t})) \right] \right\} \]  
(A.48)

and

\[ V_{it}^* = \frac{W_{it}^*}{\beta_i} - \frac{1 - \beta_i}{\beta_i} \left\{ e^{-A_i(T-t)}(y_{it}^* x_{it}) \frac{1}{\beta_i} \mathcal{N}(-\tilde{d}_it(x_{it})) - e^{-r(T-t)} W_{it} \mathcal{N}(-\tilde{d}_i(x_{it})) \right\} \]  
(A.49)

Applying Itô's Lemma on \( V_{it}^* = V_{i}(x_{it}, t) \), we obtain:

\[ dV_{it}^* = (-dt + \left( -x_{it} \frac{\partial}{\partial x_{it}} \kappa \right) dw_t). \]  
(A.50)

Equating the diffusion terms in (A.50) and (6), we obtain (35).

**Proof of Proposition 4.** Given the equilibrium default thresholds in (26), default probabilities follow from Lemma 3.

**Proof of Proposition 5.** Results follow from Lemma 3.

**Proof of Proposition 6.** Results follow from Lemma 3.

### B Benchmarks

**B.1 Benchmark (a): No Cost of Default**

\[ C_{iT} = 0 \quad \forall \ i \in 1, 2 \]  
(A.51)

**Bank i's problem.** The representative equityholder of Bank \( i \) faces the following optimization problem:

\[ \max_{W_{iT}} E[u_i(W_{iT})] \quad \text{s.t.} \quad E[x_{iT}W_{iT}] \leq W_{io} \]  
(A.52)

where the static budget constraint obtains by substituting \( V_{io} = W_{io} + D_{io} \), \( V_{iT} = W_{iT} + D_{iT} \) and \( D_{io} = E[\xi_{iT}D_{iT}] \) into \( E[\xi_{iT}V_{iT}] \leq V_{io} \).
Proposition 7. The optimal value of the equity is not affected by the leverage in place:

\[ W^n_{iT} = I_i(y^n_i \xi_T) \quad \text{where} \quad y^n_i = \left( \frac{e^{-\lambda_i T}}{W^n_{i0}} \right)^{\gamma_i} \]  
(A.53)

Bank \( i \) defaults when \( W^n_{iT} < W^n_i \):

\[ I_i(y^n_i \xi_T) < W^n_i \quad \Rightarrow \quad \xi_T > u'_i(W^n_i)/y^n_i \equiv \xi^n_i \]  
(A.54)

Proof. Special case of Proposition 2 where \( \phi = \lambda = \eta_1 = \eta_2 = 0 \).

B.2 Benchmark (b): No Systemic Cost of Default

\[ C^n_{iT} = \begin{cases} 0 & \text{if } D^n_{iT} = F_i \\ \phi + \lambda (F_i - D^n_{iT}) & \text{if } D^n_{iT} < F_i \end{cases} \quad \forall \quad i \in 1, 2 \]  
(A.55)

Bank \( i \)'s problem. The representative equityholder of Bank \( i \) faces the following optimization problem:

\[ \max_{W^n_{iT}} \mathbb{E}[u_i(W^n_{iT})] \quad \text{s.t.} \quad \mathbb{E}[\xi_T(W^n_{iT} + C^n_{iT}(W^n_{iT}))] \leq W^n_{i0} \]  
(A.56)

where the static budget constraint obtains by substituting \( V^n_{i0} = W^n_{i0} + D^n_{i0} \), \( V^n_{iT} = W^n_{iT} + D^n_{iT} + C^n_{iT} \) and \( D^n_{i0} = \mathbb{E}[\xi_T D^n_{iT}] \) into \( \mathbb{E}[\xi_T V^n_{iT}] \leq V^n_{i0} \).

Proposition 8. The optimal value of the equity is not affected by the leverage in place:

\[ W^b_{iT} = \begin{cases} I_i(y^b_i \xi_T) & \text{if } \xi_T \leq \xi^b_i \\ W^n_i & \text{if } \xi^b_i < \xi_T \leq \xi^n_i \\ I_i(y_i^b x_i \xi_T) & \text{if } \xi_T > \xi^n_i, \end{cases} \]  
(A.57)

where \( y^b_i \) solves

\[ \mathbb{E} \left[ \xi_T(W^b_{iT}(y^b_i) + C_{iT}(W^b_{iT}(y^b_i))) \right] = W^b_{i0} \]  
(A.58)

Bank \( i \) defaults when \( \xi_T > \xi^b_i \).

Proof. Special case of Proposition 2 where \( \eta_1 = \eta_2 = 0 \).

C Example of Multiple Equilibria with Homogenous Banks

This section provides an example of multiple equilibria for the case of homogenous banks. As highlighted by Corollary 1, all multiple equilibria are characterized by asymmetric optimal policies. Even though \textit{ex-ante}
identical, the two banks optimally choose different equilibrium asset allocations. Proposition 9 characterizes the equilibrium. Figure 9 illustrates.

**Proposition 9.** Consider two homogeneous banks. One of the asymmetric (multiple) equilibria can be constructed as follows:

\[
W^*_T = \begin{cases} 
  I(y^*_i \xi_T) & \text{if } \xi_T \leq \xi_i \\
  W & \text{if } \xi_i < \xi_T \leq \xi_i^* \\
  I(y^*_i x \xi_T) & \text{if } \xi_i < \xi_T \leq \xi_i^* \\
  W & \text{if } \xi_i < \xi_T \leq \xi_i^* \\
  I(y^*_i z \xi_T) & \text{if } \xi_T > \xi_i
\end{cases}
\]

\[
W^*_j_T = \begin{cases} 
  I(y^*_j \xi_T) & \text{if } \xi_T \leq \xi_j \\
  W & \text{if } \xi_j < \xi_T \leq \xi_j^* \\
  I(y^*_j x \xi_T) & \text{if } \xi_j < \xi_T \leq \xi_j^* \\
  W & \text{if } \xi_j < \xi_T \leq \xi_j^* \\
  I(y^*_j z \xi_T) & \text{if } \xi_T > \xi_j
\end{cases}
\]

(A.59)

where \(y^*_i\) and \(y^*_j\) are such that

\[
y^*_i < y^*_j < y^*_i(x^\frac{z}{x})
\]

\[
\mathbb{E} \left[ \xi_T(W^*_i(y^*_i, y^*_j) + C_T(W^*_i(y^*_i, y^*_j))) \right] = W_0
\]

(A.60)

\[
\mathbb{E} \left[ \xi_T(W^*_j(y^*_j, y^*_i) + C_T(W^*_j(y^*_j, y^*_i))) \right] = W_0
\]

(A.61)

\[
\mathbb{E} \left[ \xi_T(W^*_j(y^*_j, y^*_i) + C_T(W^*_j(y^*_j, y^*_i))) \right] = W_0
\]

(A.62)

**Proof.** We present the steps to construct the asymmetric equilibrium posited in the Proposition.

**Step 1.** From part (i) we deduce that \(y_i\) must be different from \(y_j\). If this is not the case, the thresholds \((\xi_i, \bar{\xi}_i, \tilde{\xi}_i)\) would coincides across banks, and \(W_{iT}\) would be equal to \(W_{jT}\) for any realization of the SPD at time \(T\) except for the interval \(\xi < \xi_T < \xi\), where either \(W_{iT} > W_{jT}\) or \(W_{iT} < W_{jT}\). This implies that the horizon equity of one bank would always be higher than the one of the other. Since the two banks are endowed with the same initial wealth, one of the two budget constraints can not be satisfied. Therefore, it must be that in equilibrium \(y_i\) differs from \(y_j\), and w.l.o.g. we can assume that \(y_2 > y_1\). Then, it follows that

\[
\bar{\xi}_2 < \xi_1, \quad \tilde{\xi}_2 < \tilde{\xi}_1
\]

(A.63)

**Step 2.** We impose restrictions on the thresholds \((\xi_i, \bar{\xi}_i, \tilde{\xi}_i)\) for \(i \in \{1,2\}\) to ensure that the final horizon equity profile of one bank does not dominate the one of the other. Specifically, we impose that \(\bar{\xi}_2 > \tilde{\xi}_1\), thus inducing to the following ordering:

\[
\xi_2 < \min\{\xi_2, \tilde{\xi}_2\} < \max\{\xi_2, \bar{\xi}_2\} < \xi_1 < \bar{\xi}_2 < \tilde{\xi}_2
\]

(A.64)

Without such restriction, it is straightforward to show that the final horizon equity of bank 1 would be higher than the final horizon equity of bank 2 in any possible state of the world.

**Step 3.** Finding the Nash equilibrium of the strategic game entails solving for the Nash equilibrium for a given realization of \(\xi_T\) in the seven partitions of the state space. By combining the banks’ best response functions in (22), we obtain that

- for the states \(\xi_T \leq \bar{\xi}_1\) and \(\xi_T > \tilde{\xi}_2\), the Nash equilibrium (in pure strategies) is unique and such that
$W^*_{1T} \geq W^*_{2T}$;

• for the states $\xi_1 < \xi_T \leq \bar{\xi}_2$, the Nash equilibrium (in pure strategies) is multiple and such that either $W^*_{1T} > W^*_{2T}$ or $W^*_{1T} < W^*_{2T}$. However, only $W^*_{1T} < W^*_{2T}$ can be part of the Nash equilibrium of the strategic game, otherwise the final horizon equity of bank 1 would be higher than the final horizon equity of bank 2 in all the state of the world, thus preventing the banks’ budget constraints to be simultaneously satisfied.

Therefore, for each of the following seven partitions the equilibrium final horizon equities are equal to:

<table>
<thead>
<tr>
<th>$\xi_T \leq \bar{\xi}_2$</th>
<th>$\bar{\xi}_2 &lt; \xi_T \leq \min{\xi_1, \bar{\xi}_2}$</th>
<th>$\min{\xi_1, \bar{\xi}_2} &lt; \xi_T \leq \max{\xi_1, \bar{\xi}_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\xi}_1 &lt; \xi_T \leq \bar{\xi}_2$</td>
<td>$\bar{\xi}_2 &lt; \xi_T \leq \xi_1$</td>
<td>$\xi_T &gt; \bar{\xi}_1$</td>
</tr>
</tbody>
</table>

$W^*_{1T}$ $W^*_{2T}$

<table>
<thead>
<tr>
<th>$I(y_1\xi_T)$</th>
<th>$I(y_2\xi_T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W + [I(y_1\xi_T) - W]I(\xi_T &gt; \xi_2)$</td>
<td>$W + [I(y_2\xi_T) - W]I(\xi_T &gt; \xi_2)$</td>
</tr>
<tr>
<td>$W$</td>
<td>$I(y_2\xi_T)$</td>
</tr>
<tr>
<td>$I(y_1\xi_T)$</td>
<td>$I(y_2\xi_T)$</td>
</tr>
<tr>
<td>$I(y_1\xi_T)$</td>
<td>$I(y_2\xi_T)$</td>
</tr>
</tbody>
</table>

**Step 4.** The equilibrium constructed above exists providing that a consistent set of Lagrange multipliers $(y_1, y_2)$ exists. Consistent multipliers means that they have to satisfy the restrictions imposed in **Step 1** and **Step 2**. Note that the restriction in **Step 2**, $\bar{\xi}_2 > \bar{\xi}_1$, implies that

$$\frac{\alpha}{y_2} > \frac{\alpha}{y_1} \quad \Rightarrow \quad y_2 < y_1 \left(\frac{x}{z}\right)$$

(A.65)

where $\alpha$ solves the following equation

$$\left[u(W) - u(I(\alpha))\right] \frac{1}{\alpha} + I(\alpha) - W + \phi = 0.$$  

(A.66)

Hence, the optimal Lagrange multipliers $(y_1^*, y_2^*)$ must satisfy the restriction

$$y_1^* < y_2^* < y_1^* \left(\frac{x}{z}\right)$$

(A.67)

and make the budget constraints of the two banks binding

$$E[\xi_T(W^*_{1T}(y_1^*, y_2^*) + C_T(W^*_{1T}(y_1^*, y_2^*)))] = W_0$$  

(A.68)

$$E[\xi_T(W^*_{2T}(y_2^*, y_1^*) + C_T(W^*_{2T}(y_2^*, y_1^*)))] = W_0$$  

(A.69)

$\Box$
BEST RESPONSE STRATEGIES \( \left( W_{1T}(W_{2T}), W_{2T}(W_{1T}) \right) \) and NASH EQUILIBRIA \( \left( W^*_1, W^*_2 \right) \) for a given \( \xi_T \)

\[
\begin{align*}
\xi_T \leq \xi_2 & \quad \xi_2 < \xi_T \leq \xi_1 & \quad \xi_1 < \xi_T \leq \xi_2 & \quad \xi_1 < \xi_T \leq \bar{\xi}_2 \quad \xi_2 < \xi_T \leq \bar{\xi}_1 \quad \xi_T > \bar{\xi}_1
\end{align*}
\]

Multiple Equilibria

NASH EQUILIBRIUM \( \left( W^*_1(\xi_T), y^*_1 \right) \) and \( \left( W^*_2(\xi_T), y^*_2 \right) \)

\[
\begin{align*}
W^*_1(\xi_T) & \quad W^*_2(\xi_T) & \quad W^*_1(\xi_T), W^*_2(\xi_T), p \theta(\xi_T | F_0) & \quad y_2 = \text{BC}_2(y_1), y_2 = \text{BC}_2(y_1)
\end{align*}
\]

Figure 9: Example of a multiple nash equilibrium with homogeneous banks
References


